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THE ANALYTIC PROPERTIES OF THE VACUUM EXPECTATION VALUE OF A PRODUCT OF THREE SCALAR LOCAL FIELDS

BY

G. KÄLLÉN AND A. WIGHTMAN

København 1958
i kommission hos Ejnar Munksgaard

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Synopsis.

From the general requirements of Lorentz invariance, reasonable mass spectrum shape, and local commutativity it follows that the vacuum expectation value of a product of field operators is the boundary value of an analytic function. A corresponding statement holds for the Fourier transform of the retarded commutator and for the time ordered product of the fields. For the special case of three scalar fields, it is shown that, in general, the domains of analyticity obtained in x-space and in p-space are identical. This domain is explicitly computed and shown to be bounded by pieces of analytic hypersurfaces. These surfaces intersect in corners which are of such a kind that the domain is not a natural domain of analyticity. The holomorphy envelope of this domain is computed using only elementary methods. The result turns out also to be bounded by pieces of analytic hypersurfaces.

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I. Introduction.

During the last few years, some effort has been devoted to an exploration of the mathematical structure of the vacuum expectation value of a product of local field operators. In these investigations it is customary to use only very general physical requirements of the theory. Research of this kind is made either with the objective of characterizing those properties of local, relativistic field theory that must hold whatever the nature of the interaction between the fields, or with the more practical motivation of getting tools for the handling of particular theories like quantum electrodynamics. Whatever the motivation is, the following requirements are conventionally used.

I. The theory must be invariant under Lorentz transformations.
II. The energy-momentum spectrum must be physically reasonable.
III. A field operator at a point x commutes with a field operator at a point x' if the distance between x and x' is space-like, i. e. if $(x-x')^2 > 0$.

We shall refer to condition III as the condition of "local commutativity". Condition II requires some further elaboration. From I we conclude that there exist "displacement operators" P_μ with the properties

$$[P_\mu, P_\nu] = 0, \tag{1}$$

$$[P_\mu, A(x)] = i\frac{\partial A(x)}{\partial x_\mu}. \tag{2}$$

($A(x)$ in Eq. (2) is any field that does not depend explicitly on the coordinates). From (1) it follows that we can introduce a special representation in the Hilbert space in which every state is a simultaneous eigenstate of all the P_μ's with eigenvalues p_μ. Our condition II can then be more precisely formulated in terms of these eigenvalues p_μ in the following way:

a) There exists a unique state (the vacuum) for which all components of the vector p_μ are zero.
b) For all other states, the vector p_μ is time- (or possibly light-) like with a positive value of the time component p_0.

In the special representation mentioned here, it follows from (2) that the x-dependence of any matrix element of the field $A(x)$ must be of an exponential form

$$\langle a \mid A(x) \mid b \rangle = \langle a \mid A \mid b \rangle e^{i(p^{(b)} - p^{(a)})x}. \tag{3}$$

In Eq. (3), $p_\mu^{(a)}$ and $p_\mu^{(b)}$ are the eigenvalues of the operators P_μ belonging to the states $\mid a \rangle$ and $\mid b \rangle$.

For the particular case of the vacuum expectation value of two field operators, the results are rather complete[1]. As a special case of (3) and from condition II it follows that the "frequency" of a matrix element from the vacuum $\mid 0 \rangle$ to any other state $\mid z \rangle$ must be positive

$$\langle 0 \mid A(x) \mid z \rangle = \langle 0 \mid A \mid z \rangle e^{ip^{(z)}x}, \quad \text{with} \quad p_0^{(z)} > 0. \tag{4}$$

Therefore, if we introduce a complete set of states $\mid z \rangle$ as "intermediate states" in the vacuum expectation value of two scalar fields $A(x)$ and $B(x')$ (we consider the case of scalar fields for simplicity; analogous results hold for fields with other transformation properties), we can write it in the form

$$\left.\begin{aligned} \langle 0 \mid A(x) B(x') \mid 0 \rangle &= \sum_{\mid z \rangle} \langle 0 \mid A \mid z \rangle \langle z \mid B \mid 0 \rangle e^{ip^{(z)}(x-x')} \\ \to \frac{1}{(2\pi)^3} \int dp\, e^{ip(x-x')} G^{AB}(p) &= \frac{1}{(2\pi)^3} \int dp\, e^{ip(x-x')} \Pi(p^2)\, \Theta(p), \end{aligned}\right\} \tag{5}$$

$$\Theta(p) = \frac{1}{2}\left[1 + \frac{|p_0|}{p_0}\right], \tag{5a}$$

$$G^{AB}(p) = \Pi(p^2)\,\Theta(p) = V \sum_{p^{(z)} = p} \langle 0 \mid A \mid z \rangle \langle z \mid B \mid 0 \rangle. \tag{5b}$$

(V = volume of periodicity). The function $G^{AB}(p)$ is different from zero only if the vector p lies inside the forward light cone, a fact that is clearly exhibited in Eq. (5b). It must be emphasized that the sum in (5b) goes only over those states for which the *vector* p (and not, e. g., its square) has a given value. As is easily shown, there are only a finite number of states of this kind and, therefore, the sum defining the function $G^{AB}(p)$ can never diverge. If $A(x)$ and $B(x)$ are *renormalized* fields for which the matrix elements (3) (and (4)) are finite, the computation of the function $G^{AB}(p)$ can be carried through without encountering any infinite quantities[2]).

The vanishing of the function $G^{AB}(p)$ whenever p is not within the forward light cone tells us that the function $F^{AB}(x - x')$, defined by

[1] H. UMEZAWA and S. KAMEFUCHI, Prog. Theor. Phys. **6**, 543 (1951); G. KÄLLÉN, Helv. Phys. Acta **25**, 417 (1952); H. LEHMANN, Nuovo Cimento **11**, 342 (1954); M. GELL-MANN and F. E. LOW, Phys. Rev. **95**, 1300 (1954).

[2] If the theory contains a particle with zero mass (the photon), this statement is not quite correct, as states with arbitrarily many photons of arbitrarily low frequencies cause some complications. The possible infinities encountered in this way are usually classified as "infrared". The simplest way to avoid them is to give the photon a small mass, which is the position that we adopt here. This means that all the vectors p are supposed to be time-like.

$$F^{AB}(x-x') = \langle 0 \mid A(x)\, B(x') \mid 0 \rangle, \tag{6}$$

is the boundary value of an analytic function $F^{AB}(z)$ regular for all complex numbers z that can be written in the form

$$z = -(x - x' - i\eta)^2, \tag{7}$$

with η a time-like vector in the forward light cone[3]. As is easily seen, every point that does not lie on the positive, real axis can be represented in the form (7). Therefore, $F^{AB}(z)$ is regular everywhere in the complex plane cut along the positive, real axis. (For brevity, we shall refer to this domain as "the cut plane").

If we compute the vacuum expectation value of the product $B(x')\,A(x)$, we get in an analogous way another analytic function $F^{BA}(z)$ of the same variable z which is regular for all points that can be written in the form

$$z = -(x' - x - i\eta)^2. \tag{8}$$

As (7) and (8) cover the same domain (the cut plane), $F^{AB}(z)$ and $F^{BA}(z)$ have the same domain of analyticity.

So far, no use has been made of the local commutativity. It follows immediately from III that the two functions $F^{AB}(z)$ and $F^{BA}(z)$ are equal for real, negative values of z. As they are both analytic functions they are then equal for all other values of z. From this it then follows that $G^{AB}(p) = G^{BA}(p)$. In this way, our condition III reduces the number of independent vacuum expectation values in the theory[4], but does not restrict, e. g., $G^{AB}(p)$ itself.

Frequently, one is interested not only in the expectation value of the ordinary product in (5), but also in, e. g., the "retarded commutator"[5]

$$\Theta(x-x')\langle 0 \mid [A(x),\ B(x')] \mid 0 \rangle = \frac{-i}{(2\pi)^4}\int dp\, e^{ip(x-x')}\, H(p). \tag{9}$$

As the two fields $A(x)$ and $B(x')$ commute for space-like separations, the retarded commutator vanishes except where $x - x'$ lies in the forward light cone, and its vacuum expectation value is invariant under Lorentz transformations without time reflections. Its Fourier transform $H(p)$ in (9) is therefore also the boundary value of a certain analytic function $H(z)$ regular for all points z that can be written in the form

$$z = -(p + i\eta)^2;\ \eta^2 < 0;\ \eta_0 > 0, \tag{10}$$

[3] A. Wightman, Phys. Rev. **101**, 860 (1956). Note the difference in metric between this paper and the present discussion where we have put $x^2 = \vec{x}^2 - x_0^2$.

[4] The equality $G^{AB} = G^{BA}$ can also be deduced from other hypotheses without the use of III. For example, the well-known PCT theorem implies, in particular, that in a theory invariant under strong inversion, $G^{AB} = G^{BA}$. Cf., e. g., R. Jost, Helv. Phys. Acta **30**, 409 (1957).

[5] In many applications the retarded commutator can be replaced by the "time-ordered product" $\langle 0 \mid P(A(x)\, B(x')) \mid 0 \rangle = \Theta(x-x')\langle 0 \mid A(x)\, B(x') \mid 0 \rangle + \Theta(x'-x)\langle 0 \mid B(x')\, A(x) \mid 0 \rangle$ (cf. below). Both the retarded commutator and the time-ordered product are well-defined quantities only if the commutator $\langle 0 \mid [A(x), B(x')] \mid 0 \rangle$ is not too singular for $x = x'$. For the moment, we assume this to be the case.

i. e., analytic in the cut plane. This property of $H(p)$ can also be demonstrated with the aid of the following formal calculation. Using the integral representation

$$\Theta(x) = \frac{1}{(2\,i\pi)} \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \frac{d\tau}{\tau - i\varepsilon} e^{i\tau x_0};\ \varepsilon > 0 \tag{11}$$

and

$$\langle 0 \mid [A(x),\ B(x')] \mid 0 \rangle = \frac{1}{(2\,\pi)^3} \int dp\, e^{ip(x-x')}\, \Pi(p^2)\, \varepsilon(p) \tag{12}$$

with

$$\varepsilon(p) = \frac{p_0}{|p_0|}, \tag{12a}$$

we can write

$$\left.\begin{aligned} \Theta(x-x')\langle 0 \mid [A(x), B(x')] \mid 0 \rangle &= \frac{-i}{(2\,\pi)^4} \int dp\, e^{ip(x-x')} \lim_{\varepsilon \to 0} \int \frac{d\tau}{\tau - i\varepsilon} e^{i\tau(x_0 - x_0')}\, \Pi(p^2)\, \varepsilon(p) \\ &= \frac{-i}{(2\,\pi)^4} \int dp\, e^{ip(x-x')} \lim_{\varepsilon \to 0} \int \frac{d\tau}{\tau - i\varepsilon} \Pi(\overline{p}^2 - (p_0 + \tau)^2)\, \varepsilon(p + \tau) \\ &= \frac{-i}{(2\,\pi)^4} \int dp\, e^{ip(x-x')} \lim_{\varepsilon \to 0} \overline{\Pi}(\overline{p}^2,\ p_0;\ \varepsilon), \end{aligned}\right\} \tag{13}$$

with

$$\left.\begin{aligned} \overline{\Pi}(\overline{p}^2,\ p_0;\ \varepsilon) &= \int_{-\infty}^{+\infty} \frac{d\tau\, \Pi(\overline{p}^2 - \tau^2)\, \varepsilon(\tau)}{\tau - p_0 - i\varepsilon} = \int_0^{\infty} d\tau\, \Pi(\overline{p}^2 - \tau^2) \left[\frac{1}{\tau - p_0 - i\varepsilon} + \frac{1}{\tau + p_0 + i\varepsilon}\right] \\ &= \int_0^{\infty} \frac{da\, \Pi(-a)}{a + \overline{p}^2 - (p_0 + i\varepsilon)^2}. \end{aligned}\right\} \tag{14}$$

Equation (14) shows explicitly how $H(p)$ in (9) is the boundary value of the analytic function $\overline{\Pi}(z)$

$$\overline{\Pi}(z) = \int_0^{\infty} \frac{da\, \Pi(-a)}{a - z}, \tag{15}$$

regular in the cut z-plane, with

$$z = -\overline{p}^2 + (p_0 + i\varepsilon)^2. \tag{15a}$$

An analogous calculation for the time-ordered vacuum expectation value yields the result

$$\langle 0 \mid P(A(x)\ B(x') \mid 0 \rangle = \frac{-i}{(2\,\pi)^4} \int dp\, e^{ip(x-x')} \int_0^{\infty} \frac{da\, \Pi(-a)}{a + p^2 - i\varepsilon}. \tag{16}$$

Thus, the Fourier transform of the time-ordered product is a different boundary value of the *same* analytic function $\overline{\Pi}(z)$. It is evident that, as long as one is interested only in the real parts of (14) or (16), they are interchangeable. As an example we might mention that the vacuum expectation value of the current operator in quantum electrodynamics with a weak, external field $A_\mu^{\text{ext}}(x)$ can be written as[6]

$$\left.\begin{aligned}\langle 0 | \delta j_\mu(x) | 0 \rangle &= -i \int dx'\, \Theta(x-x') \langle 0 | [j_\nu(x'), j_\mu(x)] | 0 \rangle A_\nu^{\text{ext}}(x') + \text{ren. terms}\\ &= \frac{-1}{(2\pi)^4} \int dp\, e^{ipx} \int_0^\infty \frac{da\, \Pi(-a)}{a+\vec{p}^2-(p_0+i\varepsilon)^2}\, j_\mu^{\text{ext}}(p) + \text{ren. terms.}\end{aligned}\right\} \quad (16\text{a})$$

In the special case in which the Fourier transform $j_\mu^{\text{ext}}(p)$ is different from zero only for space-like values of p^2 (i. e., when no real particles can be created by the external field), the $i\varepsilon$ in the denominator of (16 a) does not matter and (16 a) can be replaced by

$$\left.\begin{aligned}\langle 0 | \delta j_\mu(x) | 0 \rangle &= \frac{-1}{(2\pi)^4} \int dp\, e^{ipx} \int_0^\infty \frac{da\, \Pi(-a)}{a+p^2-i\varepsilon}\, j_\mu^{\text{ext}}(p) + \text{ren. terms}\\ &= -i \int dx' \langle 0 | P(j_\nu(x')\, j_\mu(x)) | 0 \rangle A_\nu^{\text{ext}}(x') + \text{ren. terms.}\end{aligned}\right\} \quad (16\text{b})$$

When real particles can be created by the external field, (16 a) must be used instead of (16 b).

The assumption that $F^{AB}(x-x')$ is not too singular at $x=x'$ is equivalent to the assumption that $\Pi(p^2)$ behaves in such a way at infinity that the integral over a appearing in (14)–(16) is convergent. Whether or not that is the case depends on the particular theory under investigation. In so-called "renormalizable" theories, it is assumed that even if the integral (15) itself does not converge, some modified version of it with a higher power of a in the denominator and a polynomial of z outside the integral sign makes sense. The exact form of this polynomial is dependent on the number of "renormalization terms" that enter into the theory. As an example we might mention that the renormalization terms indicated in (16 a) and (16 b) are of such a form that they convert the integral over a in these formulae to[6]

$$z \int_0^\infty \frac{da\, \Pi(-a)}{a(a-z)}, \qquad (16\text{c})$$

thereby improving its convergence.

In this paper, we want to generalize these results to the vacuum expectation value of three operators, using only the same very general kind of argument that has been used previously for the two-fold expectation value. As we shall see, this generalization encounters an entirely non-trivial problem when we take the step from an analytic function of one complex variable discussed above to an analytic function of three complex variables. Before entering upon these mathematical difficulties we want to stress some of the similarities that do exist between the two cases.

[6] H. Umezawa and S. Kamefuchi, Prog. Theor. Phys. **6**, 543 (1951); G. Källén, Helv. Phys. Acta **25**, 417 (1952), appendix.

II. The Analytic Properties of the Three-Fold Vacuum Expectation Value in *x*-Space.

With the aid of an argument entirely similar to that which led to Eq. (5), we find that we can write the vacuum expectation value of three scalar fields $A(x)$, $B(x')$, and $C(x'')$ as

$$\left.\begin{aligned} F^{ABC}(x-x', x'-x'') &= \langle 0 \mid A(x) B(x') C(x'') \mid 0 \rangle \\ &= \frac{1}{(2\pi)^6} \iint dp\, dp'\, e^{ip(x-x') + ip'(x'-x'')} G^{ABC}(p, p'), \end{aligned}\right\} \quad (17)$$

where the function $G^{ABC}(p, p')$ is different from zero only if both vectors p and p' lie in the forward light cone. From this it follows in the same way as before[3]) that $F^{ABC}(x-x', x'-x'')$ is the boundary value of an analytic function of the two complex vectors $x-x'-i\eta$ and $x'-x''-i\eta'$, where η and η' vary independently in the forward light cone. With the aid of the invariance of the theory under Lorentz transformations, it can further be shown that the analytic function depends only on the following three Lorentz invariant variables[7]

$$\left.\begin{aligned} z_1 &= -(x-x'-i\eta)^2, \\ z_2 &= -(x'-x''-i\eta')^2, \\ z_3 &= -(x-x''-i(\eta+\eta'))^2. \end{aligned}\right\} \quad (18)$$

Eq. (18) defines a certain domain in the (six-dimensional) space of the variables z_k. As this domain plays an important role in the following discussion, we introduce a special name for it and call it $\mathfrak{M}$. We shall study this domain in some detail in a later paragraph. For the moment we only remark that, even if each z_k *separately* can vary over its whole cut plane, it is a non-trivial problem to determine whether or not $\mathfrak{M}$ can be described as the "product" of the three cut z_k-planes.

The same argument that was used in the Introduction to show that the two functions $F^{AB}(z)$ and $F^{BA}(z)$ were the same analytic function can be used here with only trivial modifications to show that the function $F^{ABC}(z_1, z_2, z_3)$ defined with the aid of (17) is identical with the other analogous functions that are obtained with the aid of the vacuum expectation value of other permutations of the operators $A(x)$, $B(x')$, and $C(x'')$. We can, e. g., look at the function $F^{BAC}(z_1, z_2, z_3)$ defined from

$$\left.\begin{aligned} &\langle 0 \mid B(x') A(x) C(x'') \mid 0 \rangle \\ &= \frac{1}{(2\pi)^6} \iint dp\, dp'\, e^{ip(x'-x) + ip'(x-x'')} G^{BAC}(p, p') = \text{B.V.}\, F^{BAC}(z_1, z_2, z_3), \end{aligned}\right\} \quad (19)$$

(B.V. means "boundary value of")

[7] D. Hall and A. Wightman, Mat. Fys. Medd. Dan. Vid. Selsk. **31**, no. 5 (1957).

and regular in all points that can be written in the form

$$z_1 = -(x' - x - i\,\eta)^2, \tag{20 a}$$

$$z_2 = -(x - x'' - i\,\eta')^2, \tag{20 b}$$

$$z_3 = -(x' - x'' - i\,(\eta + \eta'))^2. \tag{20 c}$$

The domain in the z-space defined by (20) is exactly the same as the domain $\mathfrak{M}$. Further, if $x - x'$ is space-like, the two vacuum expectation values in (17) and (19) are the same because of the local commutativity. This means

$$\text{B.V.}\,F^{ABC}(z_1, z_2, z_3) = \text{B.V.}\,F^{BAC}(z_1, z_3, z_2) \text{ if } (x - x')^2 > 0. \tag{21}$$

The permutation of the variables z_3 and z_2 on the right-hand side of (21) occurs because of the different ordering of the points in the definitions (17) – (18) and (19) – (20). If we now look at the particular region where not only $x - x'$, but also $x' - x''$ and *every linear combination* of these two vectors are space-like, we can choose both η and η' orthogonal to the two-dimensional manifold just mentioned. We then obtain real, negative values for all the three numbers z_k both in (18) and (20), even if the η's are different from zero. Therefore, the points obtained in this way lie in the *interior* of the domains of analyticity of the two functions F^{ABC} and F^{BAC}. Further, Eq. (21) holds on this three-dimensional manifold, as the same points can also be obtained from (18) and (20) with all imaginary parts equal to zero and space-like values of the real parts. We then conclude that the function F^{ABC} is equal to the function F^{BAC} *with its two last variables permuted* wherever both functions are defined. Further, if one of these functions happens to be regular in a region where the other one was originally not defined, the former can be considered to be the analytic continuation of the other in the new region[8]. Our assumption about local commutativity therefore permits us to extend the domain of analyticity from the original domain $\mathfrak{M}$ to the union of $\mathfrak{M}$ and the domain that is obtained from it by a permutation of z_2 and z_3. Unless $\mathfrak{M}$ happens to be symmetric under this permutation which, as we shall see later, is not the case, this is a non-trivial extension. In a similar way, we find, by considering the function $F^{ACB}(z_3, z_2, z_1)$ that it is equal to $F^{ABC}(z_1, z_2, z_3)$ wherever both are defined, and that one is the analytic continuation of the other into the domain where only one of them is defined. In particular, it follows in this way that $F^{ABC}(z_1, z_2, z_3)$ is analytic also in the domain that is obtained from $\mathfrak{M}$ by a permutation of z_1 and z_3. Further, we get a similar result for, e. g., $F^{CBA}(z_2, z_1, z_3)$, but as $\mathfrak{M}$ is symmetric under the permutation of z_1 and z_2 this does not lead to any extension of the domain of analyticity of F^{ABC} itself.

Summarizing, we have found that all the six different vacuum expectation values that can be obtained from the three operators $A(x)$, $B(x')$, and $C(x'')$ are

[8] The argument given here is a slight generalization of the corresponding argument in ref. 3) for the special case $A = B = C$.

boundary values of the same analytic function $F^{ABC}(z_1, z_2, z_3)$. This function is regular in the union U of $\mathfrak{M}$ and the two domains that are obtained from it by permutations of z_3 with z_1 and z_2. Conversely, if we have a given analytic function regular in U and satisfying certain boundedness conditions[9], we can, by taking appropriate boundary values of it according to the prescriptions inherent in (18) and (20) etc., obtain all the six vacuum expectation values that can be formed out of three scalar fields. The expressions obtained in this way fulfil all our assumptions I, II, and III in the Introduction[9].

Therefore, they give a complete characterization of the properties that follow from conditions I – III.

III. The Analytic Properties of the Three-Fold Vacuum Expectation Value in *p*-Space.

As we pointed out in the Introduction, we have two analytic functions connected with the two-fold vacuum expectation value, viz. the function $F(z)$ in x-space and the function $H(z)$ in p-space. (Cf. Eqs. (6) and (9)). For the case of the three-fold expectation value, we have previously discussed the functions analogous to $F(z)$, and now we want to show that we can also define analytic functions in p-space connected with the three-fold vacuum expectation value. Consider the expression[10]

$$\left.\begin{aligned} &\Theta(x-x')\,\Theta(x'-x'')\langle 0\,|\,[C(x''),[B(x'),A(x)]]\,|\,0\rangle + \Theta(x-x'')\,\Theta(x''-x') \\ &\times\langle 0\,|\,[B(x'),[C(x''),A(x)]]\,|\,0\rangle = \frac{1}{(2\pi)^8}\iint dp\,dp'\,e^{ip(x'-x)+ip'(x''-x)}\,H^A(p,p').\end{aligned}\right\} \quad (22)$$

If the three operators A, B, and C commute for space-like separations, the left-hand side of (22) is invariant under Lorentz transformations. (Note that, for space-like separations of x' and x'', the two iterated commutators are equal because of the Jacobi identity. Therefore, the expression is unchanged when the two-time coordinates x_0' and x_0'' become equal and the two step functions $\Theta(x'-x'')$ and $\Theta(x''-x')$ change their values). Further, this expression obviously vanishes unless the two vectors $x-x'$ and $x-x''$ both lie in the forward light cone. Therefore, $H^A(p,p')$ is the Fourier transform of a function that has the same general properties as, e. g., $G^{ABC}(p,p')$ in (17) and is thus the boundary value of an analytic function $H^A(z_1, z_2, z_3)$ regular in the domain defined by

[9] Cf. L. Schwartz, Transformation de Laplace des Distributions, Med. Lunds Mat. Sem., Suppl. (1952), p. 196.

[10] The significance of expressions of the form (21) (or (9)) with $A(x)$ etc. as Heisenberg fields was first realized by H. Umezawa and S. Kamefuchi, ref. 6). Later, similar expressions with either retarded commutators or time-ordered products have been used, e. g., by G. Källén, Mat. Fys. Medd. Dan. Vid. Selsk. **27**, no. 12 (1953); M. Goldberger, Phys. Rev. **97**, 508 (1955); F. Low, Phys. Rev. **97**, 1392 (1955); H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento **1**, 205 (1955); ibid. **6**, 319 (1957); G. Mohan, Suppl. Nuovo Cimento **3**, 440 (1957).

$$z_1 = -(p - i\eta)^2, \tag{23 a}$$

$$z_2 = -(p' - i\eta')^2, \tag{23 b}$$

$$z_3 = -(p + p' - i(\eta + \eta'))^2. \tag{23 c}$$

Clearly, (23) defines the same domain $\mathfrak{M}$ as (18).

The expression (22) is completely symmetric in the operators B and C, but the operator A (and the time x_0) plays a somewhat distinguished role. In a similar way, we can build up two analogous expressions, where $B(x')$ and $C(x'')$ are singled out and define two more analytic functions H^B and H^C. If these are then expressed in terms of the *same* variables z_k that were used in H^A and defined in (23), it follows by inspection that these two new functions are analytic in the two domains that are obtained from $\mathfrak{M}$ by a permutation of z_3 with z_1 or with z_2. We thus have a situation that is somewhat analogous to the situation in x-space, except for the fact that we have no local commutativity to assure us that the three functions are equal in some common domain. However, if we investigate the algebraic structure of (22) in some detail, we find a relation that can serve as a substitute for local commutativity. Writing the step function $\Theta(x)$ as $\frac{1}{2}(1+\varepsilon(x)) = \frac{1}{2}\left(1 + \frac{x_0}{|x_0|}\right)$, we find after some algebraic manipulations

$$\left.\begin{aligned}
&\Theta(x1)\,\Theta(12)\langle 0 \,|\, [C(2), [B(1), A(x)]] \,|\, 0\rangle \\
&+\Theta(x2)\,\Theta(21)\langle 0 \,|\, [B(1), [C(2), A(x)]] \,|\, 0\rangle \\
&= \frac{1}{4}[\langle 0 \,|\, A(x)\{B(1), C(2)\} \,|\, 0\rangle + \langle 0 \,|\, B(1)\,[C(2), A(x)] \,|\, 0\rangle \\
&\quad + \langle 0 \,|\, C(2)\,[B(1), A(x)] \,|\, 0\rangle] \\
&+\frac{1}{4}\varepsilon(x1)\langle 0 \,|\, [C(2), [B(1), A(x)]] \,|\, 0\rangle + \frac{1}{4}\varepsilon(x2)\langle 0 \,|\, [B(1), [C(2), A(x)]] \,|\, 0\rangle \\
&+\frac{1}{4}\varepsilon(21)\langle 0 \,|\, [A(x), [C(2), B(1)]] \,|\, 0\rangle \\
&+\frac{1}{4}\varepsilon(12)\,\varepsilon(x1)\,[\langle 0 \,|\, A(x)\,B(1)\,C(2) \,|\, 0\rangle + \langle 0 \,|\, C(2)\,B(1)\,A(x) \,|\, 0\rangle] \\
&+\frac{1}{4}\varepsilon(x2)\,\varepsilon(21)\,[\langle 0 \,|\, A(x)\,C(2)\,B(1) \,|\, 0\rangle + \langle 0 \,|\, B(1)\,C(2)\,A(x) \,|\, 0\rangle] \\
&+\frac{1}{4}\varepsilon(1x)\,\varepsilon(x2)\,[\langle 0 \,|\, B(1)\,A(x)\,C(2) \,|\, 0\rangle + \langle 0 \,|\, C(2)\,A(x)\,B(1) \,|\, 0\rangle].
\end{aligned}\right\} \tag{24}$$

The last three lines containing two ε-functions are completely symmetric under permutations of $A(x)$, $B(1)$, and $C(2)$ and simultaneous permutations of the arguments in the sign functions. This is not true for the other terms with one or no sign functions. However, if we compute the Fourier transform of these other terms, it follows immediately from (24) and the assumption II in the Introduction that this Fourier transform is zero, unless at least one of the vectors p, p', and $p+p'$ is time-like. Therefore, in the particular region described after equation (21), where all vectors are space-like, the function $H^A(p, p')$ is equal to the Fourier transform of the three last lines in (24). If we express all three functions H^A, H^B, and H^C in terms of the same variables p, p' (and the corresponding analytic functions in terms of the same variables z), we have the relation

$$H^A = H^B = H^C \text{ for all vectors space-like.}$$

This is a relation of exactly the same kind as (21) in x-space and it can be used in the same way to ascertain that the three analytic functions are all equal and all regular in U. We then have the rather remarkable result that both in x-space and in p-space we have only one analytic function (i. e., one in each space) and that the domains of analyticity of these two functions are the same.

The last sentence above requires a slight modification if assumptions II a) and II b) are replaced by a more detailed specification of the mass spectrum of the theory. If, e. g., we require that the mass spectrum starts with two discrete states with masses m_1 and m_2 (both different from zero) and then has a continuous part above $(m_1 + m_2)$, the analyticity domain for the H-functions is somewhat bigger than the domain for the functions in x-space. No discussion of that problem is given in this paper.

For completeness, we want to mention that the Fourier transform of the time-ordered product of the three operators A, B, and C can also be expressed as a boundary value of the same analytic function $H(z)$. In places where the signs of the imaginary parts of the boundary value do not matter, the time-ordered expression can therefore again be used instead of the retarded, iterated commutator in (22).

IV. The Domain $\mathfrak{M}$ defined by Eq. (18).

We have now arrived at a point where it is essential to have a detailed and clear idea about the domain described by Eq. (18) (or (23)), i. e., we want an explicit determination of the points z_k that can be written in the form

$$\left.\begin{aligned} z_1 &= -(x - i\eta)^2, \\ z_2 &= -(y - i\eta')^2, \\ z_3 &= -(x + y - i(\eta + \eta'))^2, \end{aligned}\right\} \quad (25)$$

where x and y are two arbitrary four-vectors, while η and η' vary inside the forward light cone. While this is an entirely elementary geometric problem, a frontal attack on it can lead to considerable complication, and we shall find it convenient to use some of the information obtained in ref. 7. We recall that a continuous mapping of one region into another may carry boundary points into interior points and interior points into boundary points. It is an important simplification, in our case, that interior points of the region over which the vectors vary are mapped into interior points of the region over which the z_k vary*. (The simple example: $y = x^2$ in which the interval $-1 \leq x \leq 1$ is mapped into $0 \leq y \leq 1$ and the interior point $x = 0$ is carried into the boundary point $y = 0$ may convince the reader that this is not trivial. Besides, the discussion in Appendix I in connection with Eq. (A. 9) gives another example of a complication of this kind). Thus, in looking for boundary points, we shall limit our attention to pairs of vectors such that at least one of η and η' lies on the light cone. Furthermore, we can be sure to get every boundary point of $\mathfrak{M}$ in this way*.

The case in which η lies on the cone and η' in its interior can easily be dealt with. Then, if we are to obtain a point on the boundary, z_1 must be real, and an argument like that used in connection with Eq. (7) assures us that, if z_1 is negative, one has an interior point z_1, z_2, z_3, while, if z_1 is $\geqq 0$, one has a boundary point. (Of course, z_2 and z_3 do not vary independently over the cut planes when z_1 varies over the non-negative axis. Their range will be determined later). An analogous consideration for η in the cone and η' on it yields boundary points with $z_2 \geqq 0$.

There remains the case when both η and η' lie on the cone. For every such point, we have

$$\left.\begin{aligned} z_1 &= -x^2 + 2\,i\,x\,\eta, \\ z_2 &= -y^2 + 2\,i\,y\,\eta', \\ z_3 &= z_1 + z_2 - 2\,xy + 2\,\eta\eta' + 2\,i\,(x\,\eta' + y\,\eta). \end{aligned}\right\} \tag{26}$$

Not all points (26) lie on the boundary. As an example, we might mention the case discussed earlier where x and y and every linear combination of them are space-like. By choosing the two η's orthogonal to both x and y, we get a point where all z_k's are real and negative. The same point can also be obtained with an η^2 different from zero and slightly changed values of x^2, y^2, and $(x+y)^2$ and must therefore correspond to an internal point in the domain. Our next task is to single out the exact conditions on x, y, η, and η' for the point (26) to be on the boundary of $\mathfrak{M}$. For that purpose we write

$$\left.\begin{aligned} x &= \alpha_{11}\,\eta + \alpha_{12}\,\eta' + q, \\ y &= \alpha_{21}\,\eta + \alpha_{22}\,\eta' + q' \end{aligned}\right\} \text{ with } q\,\eta = q\,\eta' = q'\,\eta = q'\,\eta' = 0\,. \tag{27}$$

If η and η' are linearly independent, the coefficients α_{kl} in (27) and the vectors q and q' are uniquely determined. Since q and q' are both orthogonal to the two-plane spanned by the two light-like vectors η and η', they, and every linear combination of them, are space-like, i. e.,

$$q^2 > 0\,;\; q'^2 > 0\,;\; (qq')^2 - q^2\,q'^2 \leqq 0\,. \tag{28}$$

Introducing this notation into (26), we find

* This statement is a direct consequence of Lemma 3 of ref. 7 for those points where $\lambda(z) = z_1^2 + z_2^2 + z_3^2 - 2\,(z_1 z_2 + z_2 z_3 + z_3 z_1) \neq 0$. For the exceptional points where $\lambda(z) = 0$, one can easily make a direct argument.

$$\left.\begin{aligned} z_1 &= -q^2 - 2\,\alpha_{11}\,\alpha_{12}\,\eta\eta' + 2\,i\,\alpha_{12}\,\eta\eta', \\ z_2 &= -q'^2 - 2\,\alpha_{21}\,\alpha_{22}\,\eta\eta' + 2\,i\,\alpha_{21}\,\eta\eta', \\ z_3 &= z_1 + z_2 - 2\,qq' + 2\,\eta\eta'\,[1 - \alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21}] + 2\,i\,\eta\eta'\,(\alpha_{11} + \alpha_{22}). \end{aligned}\right\} \quad (29)$$

Introducing the notation $z_k = x_k + iy_k$ for $k = 1, 2, 3$ and $r = -2\,\alpha_{12}\,\alpha_{21}\,\eta\eta'$, we can eliminate all the α_{kl} between the three equations (29) and get

$$z_3 - z_1 - z_2 = r + \frac{(z_1 + q^2)(z_2 + q'^2)}{r} - 2\,qq'. \quad (30)$$

We may now consider q^2, q'^2, and qq' as arbitrarily given numbers subject only to the restriction (28). For every given set x_1, x_2, y_1, y_2, and r, it is then possible to find numbers α_{kl} such that (29) is satisfied for some value of $\eta\eta'$. As $\eta\eta'$ is arbitrary except that it must always be negative, it follows that we can consider $x_1, \ldots y_2$ and r as arbitrarily given numbers with the only restriction that the sign of r must be the same as the sign of $y_1\,y_2$. We divide the following discussion in two parts:

a) $y_1\,y_2 > 0$ and hence $r > 0$.

A particular case of (30) is obtained if we put $q^2 = q'^2 = qq' = 0$. We then get the curve

$$z_3 = z_1 + z_2 + r + \frac{z_1\,z_2}{r}; \quad 0 < r < \infty. \quad (31)$$

For fixed values of z_1 and z_2, the point z_3 describes a hyperbola when r varies from zero to infinity. This hyperbola has its center in the point $z_1 + z_2$ and one asymptote horizontal, while the other asymptote is parallel to the direction of $z_1\,z_2$. Two typical cases are illustrated in Figs. 1 and 2, where, for definiteness, we have assumed that z_1 and z_2 both lie in the upper half plane. When they are both in the lower half plane we obtain completely symmetric pictures. Fig. 1 illustrates the case when the sum of the arguments of z_1 and z_2 is smaller than π or $x_1\,y_2 + x_2\,y_1 > 0$. The asymptote then points upwards. When the sum of the arguments just mentioned becomes bigger than π or when $x_1\,y_2 + x_2\,y_1 < 0$, this asymptote rotates into the lower half plane and we get the situation illustrated in Fig. 2. We shall prove that for non-zero values of q^2, q'^2, and qq', the points (30) fill the region to the right of the curve in Fig. 1 or above the curve in Fig. 2. Any point z_1, z_2, z_3 such that z_3 is inside this domain can be obtained not only from vectors $x - i\,\eta$, $y - i\,\eta'$ with η and η' on the light cone, but also with η and η' slightly inside. Such points are interior points of the z_1, z_2, z_3 domain. Therefore, the curve (31) is the boundary of our domain in the case $y_1\,y_2 > 0$.[11] To prove this we eliminate the parameter r in Eq. (30) and write it as an equation between x_3 and y_3

[11] A result essentially equivalent to this statement and to the corresponding Eq. (40) below was first obtained by D. Hall, thesis, Princeton 1956 (unpublished). His calculations have been considerably rearranged and simplified in the above derivation. We want to thank Dr. Hall for his kind permission to use his results here.

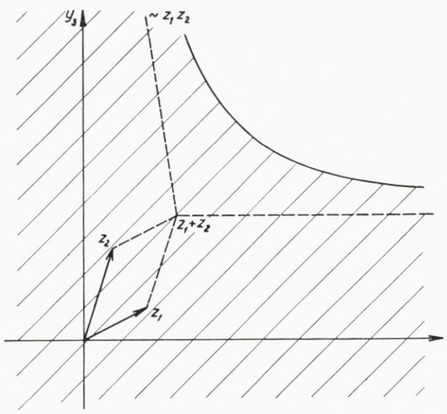

Fig. 1. The curve (31) for the case $y_1 > 0$; $y_2 > 0$; $x_1 y_2 + x_2 y_1 > 0$.

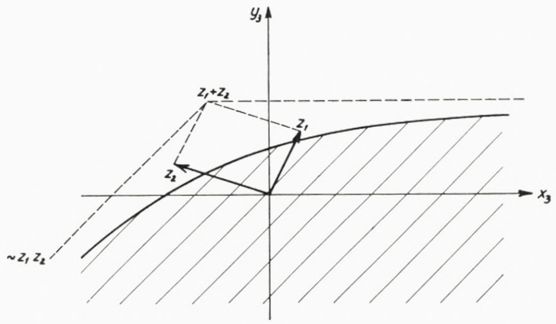

Fig. 2. The curve (31) for the case $y_1 > 0$; $y_2 > 0$; $x_1 y_2 + x_2 y_1 < 0$.

$$r = \frac{(x_1 + q^2)\, y_2 + (x_2 + q'^2)\, y_1}{y_3 - y_1 - y_2} > 0\,, \tag{32a}$$

$$x_3 = x_1 + x_2 - 2\,qq' + \frac{(x_1 + q^2)\, y_2 + (x_2 + q'^2)\, y_1}{y_3 - y_1 - y_2} + \frac{(x_1 + q^2)\,(x_2 + q'^2) - y_1\, y_2}{x_1\, y_2 + x_2\, y_1 + q^2\, y_2 + q'^2\, y_1}\,(y_3 - y_1 - y_2)\,. \tag{32b}$$

If we call $x_3^{(0)}$ the value of x_3 which we get from (32b), by putting $q^2 = q'^2 = qq' = 0$, the difference between x_3 and $x_3^{(0)}$ can be written in the following way

$$\left.\begin{aligned} x_3 - x_3^{(0)} &= -2\,qq' + \frac{q^2\, y_2 + q'^2 y_1}{y_3 - y_1 - y_2} \\ &+ (y_3 - y_1 - y_2)\, \frac{q^2 q'^2 (x_1 y_2 + x_2 y_1) + q^2 y_1 (x_2^2 + y_2^2) + q'^2 y_2 (x_1^2 + y_1^2)}{(x_1 y_2 + x_2 y_1)\,[x_1 y_2 + x_2 y_1 + q^2 y_2 + q'^2 y_1]}\,. \end{aligned}\right\} \tag{33}$$

When $x_1\, y_2 + x_2\, y_1 > 0$ (cf. Fig. 1), it follows from (32a) and $r > 0$ that we are interested only in that branch of the hyperbola for which $y_3 - y_1 - y_2 > 0$. In this case, we get from (33)

$$x_3 - x_3^{(0)} \geq 2 \left[\frac{(q^2 y_2 + q'^2 y_1)\,[q^2 q'^2 (x_1 y_2 + x_2 y_1) + q^2 y_1 (x_2^2 + y_2^2) + q'^2 y_2 (x_1^2 + y_1^2)]}{(x_1 y_2 + x_2 y_1)\,[x_1 y_2 + x_2 y_1 + q^2 y_1 + q'^2 y_2]}\right]^{\frac{1}{2}} - 2\,qq'. \tag{34}$$

With the aid of the inequality

$$\left.\begin{aligned} (q^2\, y_2 + q'^2\, y_1)\,(q^2\, y_1\, |\, z_2\, |^2 + q'^2\, y_2\, |\, z_1\, |^2) &\geq q^2\, q'^2\, [y_1^2\, |\, z_2\, |^2 + y_2^2\, |\, z_1\, |^2 \\ + 2\, y_1 y_2\, (x_1\, x_2 + y_1\, y_2)] &\geq q^2\, q'^2\, (x_1\, y_2 + x_2\, y_1)^2, \end{aligned}\right\} \tag{35}$$

the expression in the big square bracket of (34) can be simplified and we get

$$x_3 - x_3^{(0)} \geq 2 \left[\sqrt{q^2\, q'^2} - qq'\right] \geq 0\,, \tag{36}$$

which shows that the point (30) always lies to the right of the curve (31) in the case illustrated in Fig. 1. Furthermore, every point to the right of the curve (31) is obtained in this way, as one can easily convince oneself by examining (32b). When $x_1 y_2 + x_2 y_1 < 0$ (cf. Fig. 2), we are only interested in values of q^2, q'^2, and qq' so small that the inequality $(x_1 + q^2)\, y_2 + (x_2 + q'^2)\, y_1 < 0$ holds. Otherwise, it follows from (32a) that $y_3 - y_1 - y_2$ is positive and the point lies above the curve in Fig. 2. In the interesting case, we have $y_3 - y_1 - y_2 < 0$ and we see, with the aid of the inequality (35), that the last two terms in (33) are negative. It follows that

$$x_3 - x_3^{(0)} \leq -2\, qq' - 2\sqrt{q^2\, q'^2} \leq 0. \tag{37}$$

Therefore, the point (30) lies to the left of the curve (31) in Fig. 2. Again it is easy to see that (32b) yields every such point, so our statement is proved.

b) $y_1\, y_2 < 0$.

In this case, the parameter r must be negative, according to the remark made after Eq. (30). To get the boundary in this case we change our notation slightly and introduce

$$\left.\begin{aligned} \frac{q^2}{-r} &= k\,(1+\delta) \\ \frac{q'^2}{-r} &= \frac{1}{k}(1+\delta) \end{aligned}\right\} \quad k > 0;\;\; 1+\delta > 0. \tag{38}$$

Eq. (30) now becomes

$$z_3 = z_1\,(1-k) + z_2\left(1 - \frac{1}{k}\right) - \delta\left(z_1\, k + \frac{z_2}{k}\right) + r\,\delta^2 + \frac{z_1\, z_2}{r} + \varDelta, \tag{39}$$

with

$$\varDelta = -2\, qq' - 2\sqrt{q^2 q'^2} \leq 0. \tag{39a}$$

As a special case of (39), we get for $\delta \to 0$, $r \to \infty$, $\varDelta \to 0$, and $r\delta^2 \to 0$,

$$z_3 = z_1\,(1-k) + z_2\left(1 - \frac{1}{k}\right); \quad 0 < k < \infty. \tag{40}$$

This curve is also a hyperbola with its center in $z_1 + z_2$ and asymptotes along the $-z_1$ and $-z_2$ directions (cf. Fig. 3). We now want to prove that the points (39) fill the region to the left of the curve (40). We first remark that (39) for fixed finite r, δ, and $\varDelta$ and varying k defines a curve with asymptotes parallel to those of (40), but intersecting in the point $z_1 + z_2 + \frac{z_1\, z_2}{r} + r\,\delta^2 + \varDelta$ instead of $z_1 + z_2$. Because $r\,\delta^2$ and $\varDelta$ are ≤ 0 and $z_1\, z_2$ lies between the asymptotes of (40), the asymptotes of the curve (39) lie beyond those of (40). Furthermore, the curve (39) never crosses (40). To see this, one need only to assume that, for some positive k' and k,

$$z_1\,(1-k') + z_2\left(1 - \frac{1}{k'}\right) = z_1\,(1-k) + z_2\left(1 - \frac{1}{k}\right) - \delta\left(z_1\, k + \frac{z_2}{k}\right) + r\,\delta^2 + \frac{z_1\, z_2}{r} + \varDelta. \tag{41}$$

If one eliminates x_2 between the real and the imaginary parts of Eq. (41), one finds

$$y_2 = y_1 \left[\frac{(k-k')^2}{kk'} (1+\delta) + \frac{\Delta}{r} \right] \cdot \left[\left(\frac{x_1}{r} + \frac{1}{k'} - \frac{1+\delta}{k} \right)^2 + \frac{y_1^2}{r^2} \right]^{-1}, \tag{42}$$

in contradiction with the condition $y_1 y_2 < 0$. Thus, the curve (39) lies entirely within the region to the left of (40). Finally, it is easy to see that, by varying Δ holding k, δ, and r fixed, one gets from (39) along with any point z_3 all points $z_3 - \varrho$, $\varrho > 0$. By the same argument as used in case a), we conclude that all points which are to the left of the curve (40) are interior points of our domain and, therefore, (50) is really the boundary.

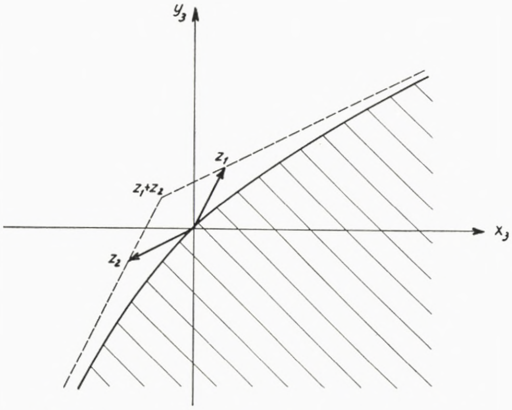

Fig. 3. The curve (40) with $y_1 > 0$, $y_2 < 0$.

At first sight it might seem a bit puzzling that we have obtained the curve (40) only by the limiting process $\delta \to 0$, $r \to \infty$, etc. The calculations of Appendix II provide an explanation; the pairs of vectors $x - i\eta$ and $y - i\eta'$ whose scalar products yield points on (40) have linearly dependent light-like vectors η and η'. In (27), we assumed η and η' linearly independent.

In the preceding discussion we have ignored the cases $y_1 = 0$, $x_1 < 0$ and $y_2 = 0$, $x_2 < 0$. This was purely a matter of convenience. These cases can be obtained by simple limiting processes from those described above. It is obviously necessary that the curves (31) and (40) then coincide, and indeed they do.

To complete the specification of the boundary we have to return to the cuts. It is now easy to give the restrictions on z_2 and z_3 when, for example, z_1 lies on the cut $z_1 = \varrho > 0$. For z_1, z_2, z_3 to lie on the boundary, z_3 has to lie on the permitted side of the curve (31) or (40), depending on whether z_2 lies on the same side of the real axis as z_1 or not.

It is worth noting that both (31) and (40) are of the form $F(z_k, r) = 0$, where F is an analytic function of the complex variables z_k and also depends on one real parameter r (or k in (40)). A surface of this kind is conventionally called an "analytic hypersurface". In the $2n$-dimensional space of the n complex z_k, this surface has the dimension $2n-1$, i. e., it is five-dimensional for our case of three complex variables. They should be distinguished from the $(2n-2)$-dimensional surfaces defined by $F(z_k) = 0$, where F is an analytic function of z_k, but does not depend on any real parameter. Surfaces of the latter kind are also of some importance for the discussion below and are called "analytic manifolds". Since the two analytic hypersurfaces (31) and (40) will be very important in the following discussion, we shall introduce special names for them, denoting (31) and (40) by F_{12} and S, respectively. The reason for these denotations will become clear from the discussion in the next paragraph.

In summary, we have found that the boundary of our domain, $\mathfrak{M}$, in z_1, z_2, z_3 space determined by (18) is composed of pieces of four analytic hypersurfaces:

The cuts
$$z_1 = \varrho \geq 0, \tag{43a}$$
$$z_2 = \varrho \geq 0, \tag{43b}$$

$$F_{12}: z_3 = z_1 + z_2 + r + \frac{z_1 z_2}{r}; \quad 0 < r < \infty, \quad \text{(relevant when } y_1 y_2 > 0) \tag{44}$$

$$S: z_3 = z_1(1-k) + z_2\left(1 - \frac{1}{k}\right); \quad 0 < k < \infty, \quad \text{(relevant when } y_1 y_2 < 0). \tag{45}$$

V. The Enlarged Analyticity Domain Following from Local Commutativity.

If we permute, e. g., z_1 and z_3 in F_{12}, we get a curve which we shall call F_{23} with the equation

$$z_1 = z_2 + z_3 + r + \frac{z_2 z_3}{r}. \tag{46}$$

To compare F_{23} with F_{12} we solve for z_3 and get

$$z_3 = r \frac{z_1 - z_2 - r}{r + z_2}; \quad 0 < r < \infty. \tag{47}$$

This curve is no longer a hyperbola. It starts at the origin with the slope $z_1/z_2 - 1$ and has a horizontal asymptote through the point z_1. It intersects the real axis in a point P with the coordinate

$$z_3 = \frac{y_1 - y_2}{y_1 y_2}(x_1 y_2 - x_2 y_1) \tag{48}$$

for

$$r = \frac{1}{y_1}(x_1 y_2 - x_2 y_1). \tag{48a}$$

Further, it intersects its own asymptote in the point

$$z_3 = \frac{y_1 x_2^2 (y_2 - y_1) - y_2 (y_2^2 - x_1 x_2)}{y_2 \quad x_1 y_2 + x_2 y_1} \tag{49}$$

for

$$r = -\frac{y_1 (x_2^2 + y_2^2)}{x_1 y_2 + x_2 y_1}. \tag{49a}$$

As the parameter r has to be positive, these intersections only take place if the imaginary part of the ratio z_2/z_1 has the same sign as y_1 in Eq. (48) and if the product $z_1 z_2$

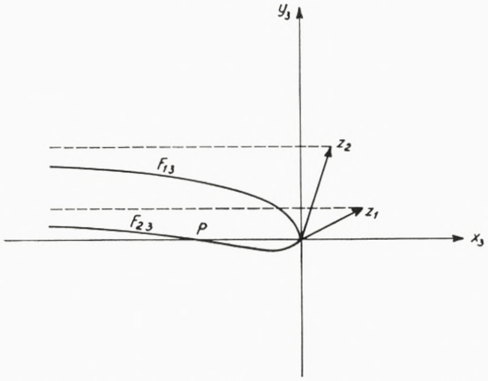

Fig. 4. The two curves F_{13} and F_{23} for $y_2 > y_1 > 0$ and $x_1 y_2 - x_2 y_1 > 0$; $x_1 y_2 + x_2 y_1 > 0$.

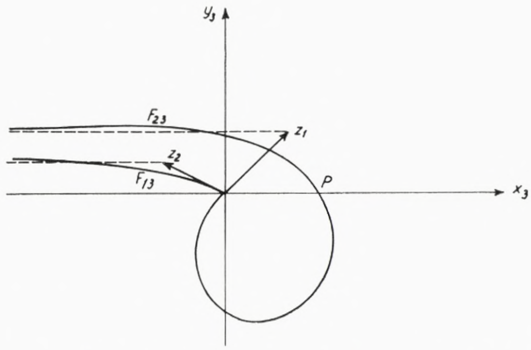

Fig. 5. The two curves F_{13} and F_{23} for $y_1 > y_2 > 0$ and $x_1 y_2 - x_2 y_1 > 0$; $x_1 y_2 + x_2 y_1 < 0$.

has an imaginary part with a sign opposite to the sign of the imaginary part of z_1 in Eq. (49). A similar curve called F_{13} is obtained from F_{12} through a permutation of z_2 and z_3. Figs. 4 and 5 illustrate a few typical cases of these curves.

If we permute, e. g., z_1 and z_3 in the S-curve, we get

$$z_1 = z_3(1-k) + z_2\left(1-\frac{1}{k}\right) \tag{50}$$

or

$$z_3 = z_1(1-k') + z_2\left(1-\frac{1}{k'}\right) \tag{51}$$

with

$$k' = \frac{k}{k-1}. \tag{51 a}$$

With the change (51 a) in the parameter, Eq. (51) is identical with the original S-curve. The new parameter k' is either negative (for $0 < k < 1$) or bigger than 1 (for $k > 1$). Apart from the change of the range of the parameter, the S-surface is therefore invariant under permutations of the z_k's. This motivates the name "the symmetric curve" or the S-curve. A complete picture of this curve with the branch corresponding to negative values of the parameter differs from Fig. 3 only by having both branches of the hyperbola in it. The S-curve intersects the real axis at the origin and also at the same point P that appears in (48). The corresponding values of the parameter k are 1 and y_1/y_2.

For completeness, we also want to mention that the curve F_{23} might have a self-intersection. This is most easily seen if we solve Eq. (47) for the parameter r. We get

$$r = \frac{1}{2}\left[z_1 - z_2 - z_3 \pm \sqrt{\lambda(z)}\right], \tag{52}$$

$$\lambda(z) = z_1^2 + z_2^2 + z_3^2 - 2 z_1 z_2 - 2 z_1 z_3 - 2 z_2 z_3. \tag{52 a}$$

If it happens that both these r-values are real and positive for a point z_1, z_2, z_3, the curve has a self-intersection in that point. This means

$$y_1 - y_2 - y_3 = 0\,, \tag{53a}$$

$$x_2\, y_3 + x_3\, y_2 = 0\,, \tag{53b}$$

$$x_2\, x_3 - y_2\, y_3 > 0\,. \tag{53c}$$

Eqs. (53) can be combined to give

$$z_3 = \left(1 - \frac{y_1}{y_2}\right) z_2^*, \tag{54}$$

$$z_3\, z_2 = \left(1 - \frac{y_1}{y_2}\right) |\, z_2\,|^2 > 0\,. \tag{54a}$$

When F_{23} really is the boundary of our domain the product $y_2\, y_3$ is positive. Hence, it follows from (54) that we must have $y_1 > y_2$, which is obviously inconsistent with (54a). Therefore, this self-intersection is never relevant for our discussion. With suitable permutations of the z's, a similar discussion is also valid for the curve F_{13}.

We are now able to draw a picture of the domain that is obtained from the domain in the previous paragraph when we permute z_1 and z_3. A few typical cases are shown in Figs. 6–8. When z_2 and z_3 are in the same half plane, F_{23} is the boundary; when they are in opposite half planes, S is the boundary. To be able to compare this

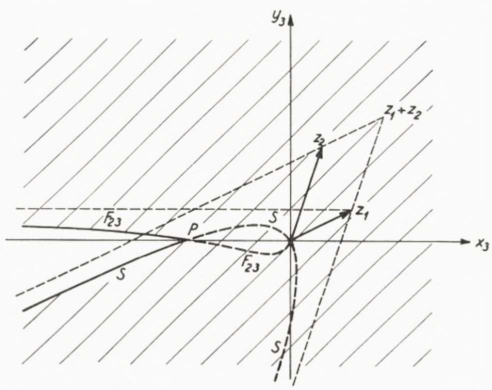

Fig. 6. The analyticity domain (unshaded) obtained after a permutation of z_1 and z_3. $y_2 > y_1 > 0$; $x_1\, y_2 - x_2\, y_1 > 0$; $x_1\, y_2 + x_2\, y_1 > 0$.

domain with the domain obtained before the permutation, we have plotted all curves in the z_3-plane for fixed values of z_1 and z_2. Further, the domain where the function might have singularities has been shaded in these pictures. It must be remarked that in those cases where part of the positive, real axis lies outside the shaded domain, that part forms a "prong" where the function might have singularities. This follows immediately from the concluding remark in the previous paragraph. Obviously, entirely similar pictures are obtained if we plot the domain we get by interchanging z_2 and z_3 in (18).

We now have to form the union of the original domain and the two domains obtained after the permutations. The simplest case happens when z_1 and z_2 are in

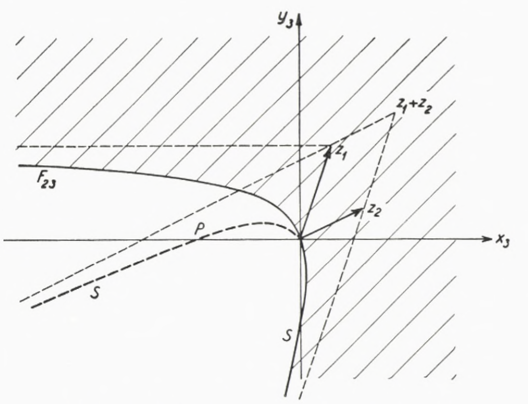

Fig. 7. $y_1 > 0$; $y_2 > 0$; $x_1 y_2 - x_2 y_1 < 0$.

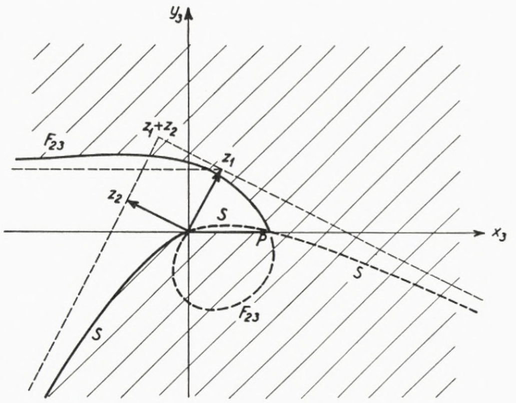

Fig. 8. $y_1 > y_2 > 0$; $x_1 y_2 - x_2 y_1 > 0$.

opposite half-planes (for definiteness, we suppose, $y_1 > 0$, $y_2 < 0$) and when $x_1 y_2 - x_2 y_1 > 0$. We then get the union of the two unshaded domains in Figs. 9 and 3, i.e., the whole z_3-plane except the positive, real axis (Figs. 9 and 10 show the union of the two domains that are obtained after permutations of z_1 and z_3 and of z_2 and z_3 when z_1 and z_2 are in opposite half-planes). This case is so simple that no special picture is required. When $x_1 y_2 - x_2 y_1$ becomes negative, we get instead the union of Fig. 10 and a figure similar to Fig. 3. The result is shown in Fig. 11. The positive, real axis

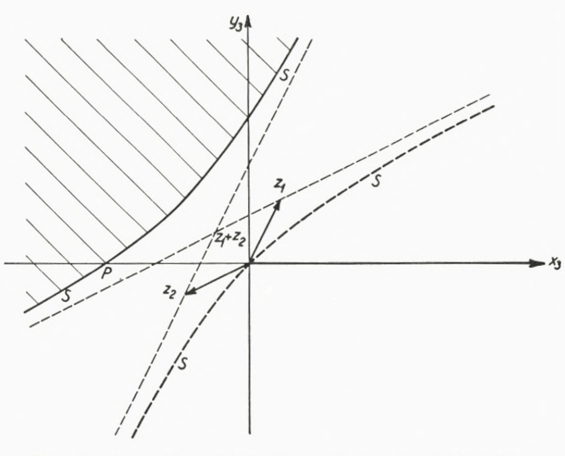

Fig. 9. The union of the domains obtained after permutations of z_1 and z_3 and of z_2 and z_3 for $y_1 > 0$; $y_2 < 0$; $x_1 y_2 - x_2 y_1 > 0$.

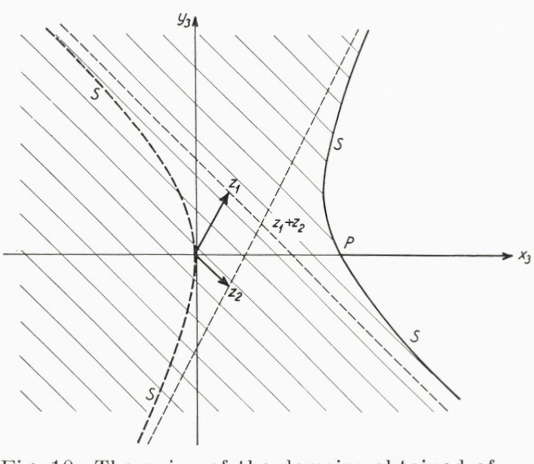

Fig. 10. The union of the domains obtained after permutations of z_1 and z_3 and of z_2 and z_3 for $y_1 > 0$; $y_2 < 0$; $x_1 y_2 - x_2 y_1 < 0$.

and the domain situated between the two branches of the S-curve are outside the domain of analyticity for the functions $F(z_k)$ and $H(z_k)$. When z_1 and z_2 lie in the same half-plane (y_1 and y_2 both positive, e. g.) we get the union of the domain in Fig. 1, the domain in one of the Figs. 6–8, and three similar pictures with F_{23} replaced by F_{13}. When the point P lies to the left of the origin, the upper of the two curves

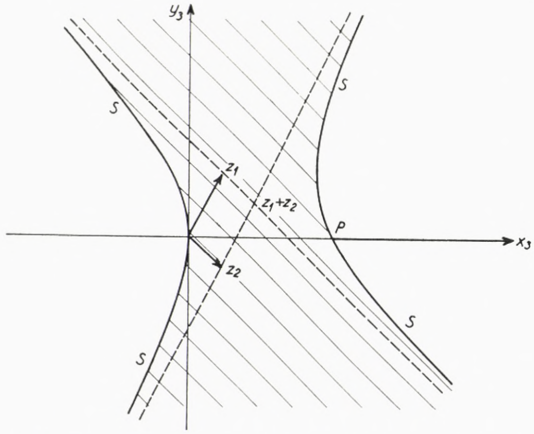

Fig. 11. The final domain for $y_1 > 0$; $y_2 < 0$; $x_1\, y_2 - x_2\, y_1 < 0$.

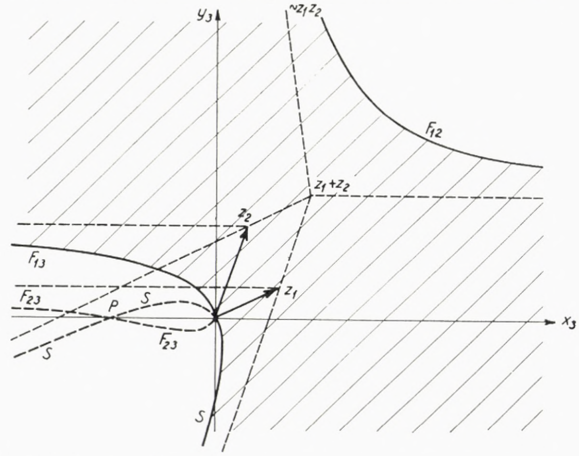

Fig. 12. The final domain for $y_2 > y_1 > 0$; $x_1\, y_2 + x_2\, y_1 > 0$; $x_1\, y_2 - x_2\, y_1 > 0$.

F_{23} and F_{13} goes to the origin (cf. Fig. 4). Therefore, the final domain is as in Fig. 12. In this two-dimensional picture the domain consists of two parts, one above the curve F_{12} and one around the negative, real axis and bounded by the upper one of F_{23} and F_{13} and by the S-curve. When the point P lies to the right of the origin, the boundary goes through the point P and has a prong along the real axis down to the origin. A more important qualitative change that might happen in Fig. 12 occurs

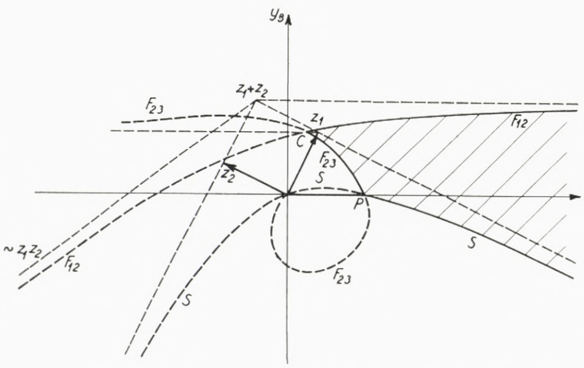

Fig. 13. The final domain for $y_1 > y_2 > 0$; $x_1\, y_2 + x_2\, y_1 < 0$; $x_1\, y_2 - x_2\, y_1 > 0$.

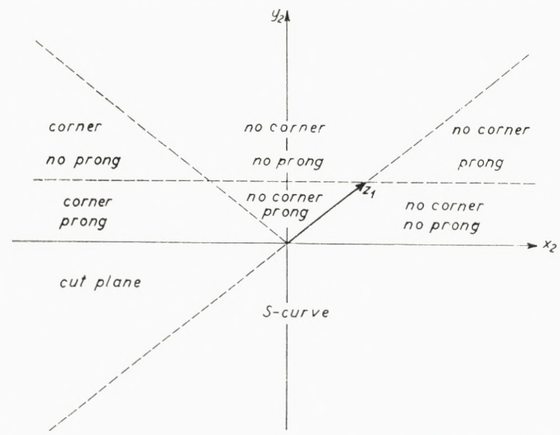

Fig. 14. General summary of the final domain. The figure indicates the position of z_2 relative to z_1 for the different cases to happen.

when we have $x_1 y_2 + x_2 y_1 < 0$. In that case we get Fig. 13, where the domain of analyticity is connected also in the two-dimensional picture. At the same time, a "corner" between the two curves F_{12} and F_{23} has appeared in the point C with the coordinate given by (49). In Fig. 13, we have also put the point P to the right of the origin. This is independent of the appearance of the corner, and we have four possible combinations depending on the relative position of z_1 and z_2. A survey of the different possibilities is given in Fig. 14.

We have now obtained a complete description of the domain of analyticity for our functions. A very characteristic feature of our result is the appearance of the corners or intersections between the different analytic hypersurfaces that appear. We have already mentioned one of them in Fig. 13, but also want to point out that there is another corner in our domain connected with Fig. 11. When z_1 and z_2 are collinear (i. e. $x_1 y_2 - x_2 y_1 = 0$), the S-curve degenerates into a straight line through z_1, z_2, and the origin. This line is a self-intersection of the S-boundary and also forms a corner in our domain. In the discussion below, these different corners are of paramount importance.

VI. Analytic Completion and Natural Domains of Analyticity for Functions of More than One Complex Variable.

It is an important phenomenon in the theory of analytic functions of more than one complex variable that an arbitrary domain in the $2n$-dimensional space of n complex numbers cannot be a domain of analyticity for an analytic function. Therefore, it is in general possible to continue *every* analytic function regular in a given domain into a somewhat larger domain. This larger domain is called "the envelope of holomorphy" of the given domain, and the process of continuing a given function is conventionally called "analytic completion". A domain that is equal to its own holomorphy envelope is called a "natural domain of analyticity". Although these concepts are well known and extensively treated in the mathematical literature, they have found very few applications in physics so far and cannot be considered standard tools for theoretical physicists. For the convenience of the reader, we therefore want to give a summary of the basic concepts and methods in this field as far as they have applications in the physical problem under investigation. In this discussion, we do not try to achieve complete mathematical rigour or generality. For that purpose we have to refer to the mathematical text books[12].

We start by considering the following, much simplified problem[13]. Suppose we

[12] H. Behnke and P. Thullen, Ergebn. d. Mathem. **3**, Nr. 3, Berlin (1934) and S. Bochner and and W. T. Martin, Several Complex Variables, Princeton (1948). A corresponding phenomenon does not happen for a function of only one complex variable. In that case, it is possible to construct a function which is analytic in an arbitrarily given domain and has singularities at every point of the boundary. Cf. L. Bieberbach, Lehrbuch der Funktionentheorie I, Berlin 1930, p. 297.

[13] Cf. the book by Bochner and Martin in ref. 12). p. 64ff.

have a function $F(z_1, z_2)$ of two complex variables z_1 and z_2, analytic for all values of the z's except the point $z_1 = z_2 = 0$. We then form the integral

$$I = \frac{1}{2\pi i}\int_C \frac{dt\, F(t, z_2)}{t - z_1} \tag{55}$$

over a path C parallel to the z_1-plane. This path is illustrated in Fig. 15. As an example, we can think of C as being the unit circle for t and independent of z_2. It is evident that the integral I is an analytic function of both z_1 and z_2 and that $I = F$ for z_2 different from zero. However, we might move the path C down to the z_1-plane in Fig. 15 and I is still an analytic function of z_1 regular also for $z_1 = 0$. As $z_2 = 0$ when C lies in the z_1-plane, the integral I is an analytic function of the two z's regular also at the origin and coinciding with the given function for $z_2 \neq 0$. Therefore, it provides an analytic continuation of the given function outside the domain where it was originally defined, i. e., in the point $z_1 = z_2 = 0$. The same technique can be used to continue a given function into any isolated domain around the origin. Further, in the situation depicted in Fig. 16, where the function $F(z_1, z_2)$ is known to be analytic everywhere above the z_1-plane and outside the horn protruding from it, an integral along the path C in the same way provides an analytic continuation of F into the horn.

There are two essential features of these examples that make the application of this technique succesful, viz., that we can find a path of integration for the integral in (55) *parallel to the z_1-plane* and that we can *displace* this path in such a way that *all points inside the path are regular points.* The analytic completion can proceed as long as the *path* C lies entirely inside the original domain of analyticity. In the example in Fig. 16, we therefore get the result that our function is analytic at all points above the z_1-plane. The plane is thus the holomorphy envelope of the given domain.

From this discussion it should be clear that certain convexity properties of the given domain decide how far an analytic completion can be carried out. If, e. g., we have a function that is regular in the domain *inside* the horn in Fig. 16 and above the z_1-plane, it would not have been possible to make any analytic completion for that domain. Only very special surfaces have the property that they can be boundaries from both sides. An example of such a surface in Fig. 16 is the z_1-plane which can obviously be the boundary of a domain of holomorphy from either side.

The situation outlined above appears to be comparatively simple, but a complication is introduced because we might make an analytic transformation of our variables z_k to new variables z_k'. It is quite possible, and usually the case, that a situation which is simple before the analytic transformation is so complicated after it that it is not easy to decide off-hand where the holomorphy envelope lies. On the other hand, the opposite might also be the case and it might happen that, with the aid of a suitable transformation, a complicated domain can be cast into a form where a direct application of the technique just described is possible. In fact, this is the way in which we compute holomorphy envelopes in this paper. Roughly speaking, we thus have the situation that, if it is possible to find some analytic transformation such that the boundary of our domain has the convexity properties indicated in Fig. 16 after the transformation has been made, then it is possible to make an analytic com-

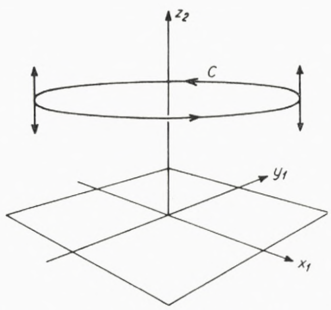

Fig. 15. The path C in Eq. (55).

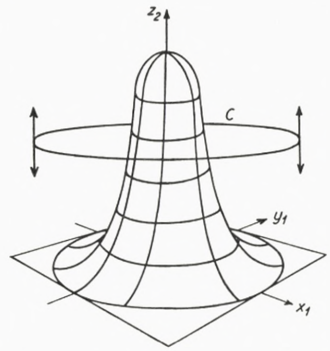

Fig. 16. Analytic completion into a "horn" for a function of two complex variables.

pletion of the functions involved. A further elaboration of this statement leads to the introduction of the notion "pseudoconvexity" of surfaces[14]. This is a property of the surface that is invariant under analytic transformations. The sign occurring in the

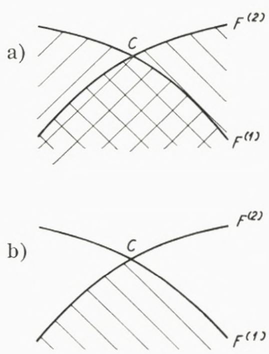

Fig. 17. The shaded parts are outside the analyticity domains.

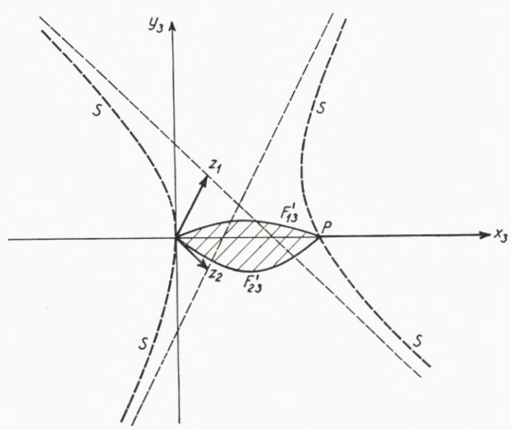

Fig. 18. Analyticity domain for the function $H(z)$ obtained from a perturbation theory calculation of the vertex part.

pseudoconvexity condition decides from which side the surface can be the boundary of a domain of analyticity. In particular, only those surfaces for which the pseudoconvexity inequality becomes an equality can be boundaries from both sides. We do not want to enter into a detailed discussion of the notion of pseudoconvexity here, but mention the following result which is important in our applications. *An analytic*

[14] Cf. BEHNKE and THULLEN, ref. 12), pp. 27—32.

hypersurface, that is a $2n-1$ dimensional manifold defined by $r=F(z_k)$, where r is a real parameter and F an analytic function of the z's, is pseudoconvex from both sides. By making an analytic transformation and introducing r as one of the variables, the hypersurface is brought into the form $\text{Im}(r)=0$, which is a plane through which it is impossible to make an analytic completion. As the boundary of our domain in the previous paragraph is made up of pieces of analytic hypersurfaces, this result tells us that the only points where it might be possible for us to continue our functions in general are the corners mentioned at the end of the last paragraph. This leads us to quote the following two results. Suppose we have two domains that are bounded by two analytic hypersurfaces. Suppose further that two of these surfaces intersect and form a "corner" of $2n-2$ dimensions. We then have (i) The domain formed by the *intersection* of the two original domains is a natural domain of analyticity (cf. Fig. 17a) and (ii) the domain formed by the *union* of the two given domains is *not* a natural domain of analyticity, *unless* the corner is an analytic manifold (cf. Fig. 17b). The proof of (i) is essentially trivial as we can take the sum of two functions, each one with a singularity on one of the given hypersurfaces, and thereby get a function that is regular in the intersection of the two given domains, but cannot be continued beyond this intersection. Therefore, the intersection of the two given domains is its own holomorphy envelope. The proof of (ii) is somewhat more complicated. An elementary argument has, e.g., been given by Kneser[15]. We do not repeat his proof here, but mention that it makes use of exactly the tools described above, viz., analytic transformations with the purpose of bringing the corner into such a shape that a direct application of integrals like (55) is possible in the new variables. In fact, it was after having studied Kneser's paper carefully that we decided to try this technique in our problem.

VII. Analytic Completion through the Corner Formed by the Self-Intersection of the S-Boundary.

We now return to the discussion of our problem and first direct our attention to the corner formed by the S-curve in the limiting case of Fig. 11, when z_1 and z_2 are collinear. This is a corner of the kind illustrated in Fig. 17b, and we are thus able to continue our functions through that corner. However, the corner is not in such a shape that it is possible to apply our technique with integrals like (55) immediately and get simple results. In this situation, it is informative to ask what simple examples calculated in perturbation theory can teach us about possible singularities. It is not very difficult to compute both the function $F(z_k)$ and the function $H(z_k)$ in first, non-trivial order of perturbation theory for the case of the so-called "vertex function". As is shown in Appendix III, one then gets the result that the function $F(z_k)$ in this approximation is analytic in the product of the three cut planes[16]. On the other

[15] H. Kneser, Math. Ann. **106**, 656 (1932).

[16] The higher order perturbation theory expressions for the vertex function have been studied by Y. Nambu, Nuovo Cimento **6**, 1064 (1957).

hand, the function $H(z_k)$ does have non-trivial singularities. From our result in the Appendix it follows that the perturbation theory expression for $H(z_k)$ can have a singularity anywhere inside the shaded domain in Fig. 18. The equations of the curves F'_{kl} are the same as the equations for the curves F_{kl} in (31), except that the parameter r now has *negative* values. From this result we immediately conclude that the holomorphy envelope we are looking for cannot lie beyond the curves F'_{kl}, and there is the possibility that the two F'_{kl}-curves are in the boundary of the envelope. Actually, this is the case, which we now proceed to prove.

We first remark that if we, e. g., put z_1 in the upper half-plane and z_2 and z_3 in the lower half-plane, the six-dimensional region bounded by the three cuts (note that each cut is a five-dimensional analytic hypersurface in our six-dimensional space) and the two branches of the S-curve, and inside which we might *a priori* have singularities, is entirely separated from all other regions with singularities. Besides, there are five other similar domains which are obtained either by putting z_1 in the lower half-plane and/or by permuting the z_k's among each other. When we know the holomorphy envelope of one of these domains we get it for the others by simple permutations.

As we have a conjecture about what the holomorphy envelope is, it is reasonable to make an analytic transformation and *introduce the parameter of the expected answer as one of the new variables.* If the conjecture is correct, we shall get a situation similar to Fig. 16 on one side of the real axis of the new variable and be able to do the continuation without difficulty. Actually, it is also convenient to introduce another new variable to simplify the S-curve a bit and write

$$\left.\begin{aligned} z_2 &= u - r, \\ z_3 &= \frac{r}{u}(z_1 - u). \end{aligned}\right\} \tag{56}$$

The inverse of this transformation is

$$\left.\begin{aligned} r &= \frac{1}{2}\left[z_1 - z_2 - z_3 + \sqrt{\lambda(z)}\right], \\ u &= \frac{1}{2}\left[z_1 + z_2 - z_3 + \sqrt{\lambda(z)}\right]. \end{aligned}\right\} \tag{56a}$$

When r is negative and real we are, according to (52), on the analytic hypersurface F'_{23}. (The quantity $\lambda(z)$ is defined in Eq. (52a)). Because of the appearance of the square root in (56a) this transformation is not one-to-one. For every point in the z-space we have two points in the space of z_1, u and r. This means that our six singularity domains mentioned before are mapped on twelve domains in the new space. All these domains are still separated from each other. This can be seen if we make the mapping unique, e. g., by introducing a cut along the hypersurface $\lambda(z) = \varrho$ with ϱ a positive, real number and defining the square root to have a positive, imaginary part. Our statement then follows from the fact that the cut lies entirely inside the original domain of analyticity and never touches the singular regions[17]. Therefore, the mapping is locally one-to-one in a neighbourhood of the singular domains.

The transformation (56) and (56a) is constructed to treat the case when z_1 lies in one half-plane and z_2 and z_3 in the other. For definiteness, consider the case in Fig.

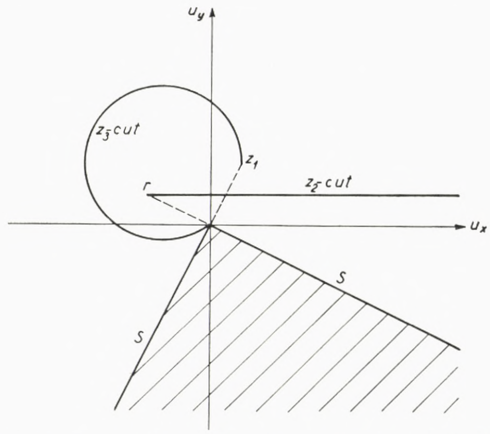

Fig. 19. The u-plane after the transformation (56).

19. This figure shows the complex plane of the variable u for fixed values of z_1 (in the upper half-plane) and r (also in the upper half-plane). The analytic hypersurface corresponding to the cut in the z_2-plane (i. e. $z_2 = \varrho$, ϱ real and positive) intersects the u-plane along a horizontal line to the right of the point r. The z_3 –cut corresponds to a circular arc between the origin and the point z_1 and with the slope rz_1 at the origin. (The rest of the circle corresponds to the negative, real axis in the z_3-plane). As z_1 is in the upper half-plane, we are interested only in the case when z_2 and z_3 lie in the lower half-plane, i. e., in that part of the u-plane in Fig. 19 that lies below the horizontal line through r and outside the circle.

[17] From (52), it follows that $\lambda(z) = \varrho$ never intersects any of the F's except in the self-intersection, which we have shown never to be on the boundary of our domain. In a similar way, it is seen that $\lambda(z) = \varrho$ never intersects the S-curve, except when

$$(z_3 =)\; z_1 + z_2 \pm \sqrt{\varrho + 4\, z_1 z_2} = z_1 (1 - k) + z_2 \left(1 - \frac{1}{k}\right)$$

or

$$z_1 = \frac{z_2}{k^2} \pm \frac{\sqrt{\varrho}}{k}.$$

Therefore, z_1 and z_2 have the same sign for their imaginary parts. As both $\lambda(z)$ and the S-curve are invariant under permutations of the z's, it follows that all z's must lie in the same half-plane for $\lambda(z) = \varrho$ to intersect the S-curve. In that case, however, the S-curve is not the boundary, which proves that $\lambda(z) = \varrho$ lies entirely on one side of the boundary. As these points trivially lie inside the domain of analyticity, e. g., when z_1 is on the negative real axis, it follows that the whole manifold lies entirely inside the domain of analyticity.

A simple calculation yields the result that, in terms of z_1, u, and r, the S-curve in (40) $(-\infty < k < \infty)$ is given by

$$\left.\begin{aligned} u &= z_1 k, \\ \text{and} \qquad u &= \frac{r}{1-k}, \end{aligned}\right\} \qquad (57)$$

i. e., two straight lines through the origin and the points z_1 and r, respectively. According to what has been said above, only those parts of these lines that lie in the lower half-plane are relevant for our discussion.

When the point r lies on the line through z_1 and the origin, the two lines representing the S-curve coincide. As is nearly trivial and also easily checked with the aid of (56), this corresponds exactly to the limiting case of Fig. 11 when z_1 and z_2 are collinear, i. e., to the corner in the S-curve. When r moves to the right of this line, we see by inspection from (56) that a point with u on the relevant part of the S-line through z_1 has $\mathrm{Im}\,(z_2/z_1) > 0$. According to Fig. 14, such points are inside the original domain of analyticity. As the S-curve is the only possible boundary in the region we are considering, it follows that all relevant points in the u-plane (i. e. outside the circle and below the horizontal line through r) are regular points. In a similar way it is seen that when r moves to the left of the line through z_1, those points in the u-plane that lie between the two S-lines are *outside* the domain of analyticity. By changing r in this way we are hence able to move our u-plane from a position where essentially every interesting point is inside the given domain to a position where a certain region (bounded by the two S-lines) is outside the domain. This corresponds roughly to the situation in Fig. 16 when the plane with the path C in it is moved from above the peak of the horn and downwards. The only difference lies in the fact that the "singular region" in the u-plane is not isolated because it is open at infinity and tied to the circle representing the z_3-cut at the origin. It is therefore not possible to introduce a closed path of integration in the way we want it. However, this feature of our transformation can be improved if we introduce infinitesimal "curvature terms" in the following way:

$$\left.\begin{aligned} u &= t + \frac{\varepsilon}{t}, \\ r &= r^{(0)}\left(1 + \varepsilon_1 t + \frac{\varepsilon_2}{t}\right), \\ z_1 &= z_1^{(0)}\left(1 + \varepsilon_3 t + \frac{\varepsilon_4}{t}\right), \end{aligned}\right\} \qquad (58)$$

where ε, $\varepsilon_1, \ldots \varepsilon_4$ are infinitesimal, (complex) numbers. If t is not very near the origin or very large, the t-plane practically coincides with the u-plane with r and z_1 in the positions $r_1^{(0)}$ and $z^{(0)}$. Therefore, we have essentially the same picture as in Fig. 19, except at the origin and very far out, but these are exactly the two regions where we want to change the character of the excluded domain a little. The line $u = -\varrho\, r$ (ϱ real and positive) becomes, after the transformation (58),

$$t + \frac{\varepsilon}{t} = -\varrho\, r^{(0)} \left[1 + \varepsilon_1\, t + \frac{\varepsilon_2}{t} \right]. \tag{59}$$

For large values of $|\,t\,|$ we can neglect the terms with ε and ε_2 in (59). The curve (59) then becomes

$$t = \frac{-\varrho\, r^{(0)}}{1 + \varrho \varepsilon_1\, r^{(0)}}, \tag{60}$$

which is a circle starting from the origin with the slope $-r^{(0)}$ and ending at the point $-1/\varepsilon_1$. With the same approximation the curve $u = -\varrho\, z_1$ becomes

$$t = \frac{-\varrho\, z_1^{(0)}}{1 + \varrho \varepsilon_3\, z_1^{(0)}}, \tag{61}$$

i. e., a circle between the origin and the point $-1/\varepsilon_3$. If we choose ε_1 real and positive and ε_3 real and negative, the two end points of the two circles both lie on the real axis and on opposite sides of the origin. Therefore, they must intersect in some point with a very large absolute value of t. The domain bounded by the two circles and lying between the origin and the other intersection is then certainly outside the domain of analyticity. To decide what happens on the other side of the intersection we remark that the z_1-cut now appears in our picture as a line from the point $-1/\varepsilon_3$ with the slope $1/z_1^{(0)}$, while the continuation of this line on the other side of $-1/\varepsilon_3$ corresponds to the negative, real z_1-axis. In a similar way, the z_3-cut corresponds to the curve

$$t^2\, r^{(0)}\, \varepsilon_1 \left[z_1^{(0)}\, \varepsilon_3 - 1 \right] + t \left[r^{(0)}\, z_1^{(0)} \left(\varepsilon_1 + \varepsilon_3 \right) - \varrho - r^{(0)} \right] + r^{(0)}\, z_1^{(0)} = 0\,. \tag{62}$$

In general, this is a somewhat complicated curve, but, as ε_1 and ε_3 are infinitesimal, it is easily seen that this curve consists of two parts, one of which is practically the circle in Fig. 19 and the other a straight line from the point $-1/\varepsilon_1$ with the slope $-1/r^{(0)}$. Again, the continuation of this line on the other side of the point $-1/\varepsilon_1$ corresponds to the negative, real z_3-axis. As the negative, real axis of any of the z's is inside the analyticity domain, and as the S-curve is the only possible boundary in the region we are considering, it follows that the region on the other side of the intersection of the two circles lies inside the domain of analyticity for our function. The situation for large absolute values of t is illustrated in Fig. 20. After the transformation (58) it is now possible to introduce a path of integration that is closed at infinity.

Very near the origin we have, instead, to consider the terms with ε, ε_2, and ε_4 in (58). The curve (59) is then approximated by

$$t + \frac{\varepsilon}{t} = -\varrho\, r^{(0)} \left[1 + \frac{\varepsilon_2}{t} \right]. \tag{63}$$

In general, this is a somewhat complicated curve, but the special case $\varepsilon = -\varepsilon_2^2$ is simple. The curve then reduces to the straight line

$$t = -\varrho\, r^{(0)} - \varepsilon_0, \tag{64}$$

$$\varepsilon_2 = -\varepsilon_0;\ \varepsilon = -\varepsilon_0^2, \tag{64 a}$$

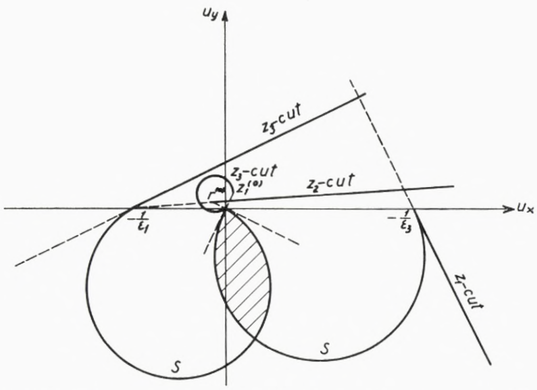

Fig. 20. The t-plane after the transformation (58). This figure shows the behaviour for large, absolute values of t.

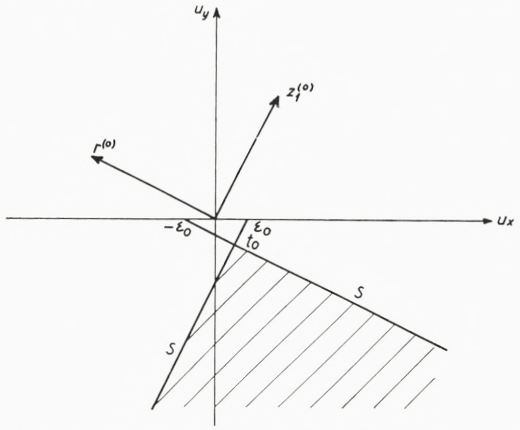

Fig. 21. The t-plane after the transformation (58). This figure shows the behaviour for small absolute values of t.

(and the point $t = -\varepsilon_2$). In a similar way, we find for $\varepsilon_4 = -\varepsilon_2 = \varepsilon_0$ that the curve $u = -\varrho\, z_1$ becomes the straight line

$$t = -\varrho^{(0)} z_1 + \varepsilon_0. \tag{65}$$

If we choose ε_0 real and positive, we get the configuration shown in Fig. 21. The two lines intersect in a point t_0 determined from

$$t = -\varrho\, z_1^{(0)} + \varepsilon_0 = -\varrho'\, r^{(0)} - \varepsilon_0. \tag{66}$$

Writing for simplicity

$$\left.\begin{aligned} z_1^{(0)} &= x_1 + i y_1, \\ r^{(0)} &= r_x + i r_y, \end{aligned}\right\} \tag{67}$$

we find from (66)

$$\varrho = \varrho' \frac{r_y}{y_1} = \frac{2\,\varepsilon_0\, r_y}{x_1 r_y - y_1 r_x} > 0. \tag{67 a}$$

As might be expected, the point t_0 is a point on the self-intersection of the S-curve. This can be proved if we compute the quantities u, r, and z_1 from (58). At the point t_0 we find

$$\left.\begin{aligned} u &= \varrho\varrho' \frac{z_1^{(0)} r^{(0)}}{t_0}, \\ r &= -\varrho \frac{z_1^{(0)} r^{(0)}}{t_0}, \\ z_1 &= -\varrho' \frac{z_1^{(0)} r^{(0)}}{t_0}. \end{aligned}\right\} \tag{68}$$

These three numbers lie on the same straight line through the origin and, according to the remark made in the paragraph after Eq. (57), this is a point on the self-intersection of the S-curve. If we consider a point $t = t_0 + \delta$ with δ a complex number of the form $\alpha\, z_1^{(0)} + \beta\, r^{(0)}$, where α and β are real numbers, we find

$$\left.\begin{aligned} \frac{z_1}{r} = \frac{z_1^{(0)}}{r^{(0)}} \frac{t+\varepsilon_0}{t-\varepsilon_0} = \frac{z_1^{(0)}}{r^{(0)}} \frac{t_0+\varepsilon_0+\delta}{t_0-\varepsilon_0+\delta} \approx \frac{\varrho'}{\varrho}\left[1+\frac{\delta}{t_0+\varepsilon_0}-\frac{\delta}{t_0-\varepsilon_0}\right] \\ = \frac{\varrho'}{\varrho}\left[1-\frac{2\,\varepsilon_0}{\varrho\varrho'}\left(\alpha\frac{1}{r^{(0)}}+\beta\frac{1}{z_1^{(0)}}\right)\right]. \end{aligned}\right\} \tag{69}$$

Therefore

$$\operatorname{Im}\left(\frac{z_1}{r}\right) = \frac{2\,\varepsilon_0}{\varrho^2}\left(\alpha\frac{r_y}{|\,r^{(0)}\,|^2}+\beta\frac{y_1}{|\,z_1^{(0)}\,|^2}\right). \tag{69 a}$$

For positive values of α and β, i. e., for a point t above the point t_0 in Fig. 21 and between the two lines representing the S-curve, $\operatorname{Im}(z_1/r) > 0$, and this point is inside the domain of analyticity. The only region in Fig. 21 which is not inside the domain of analyticity is then the domain between the two lines and below the point t_0. This shows that we can close our path of integration also for small values of t and then carry through the analytic completion according to what was said before.

The analysis given so far in this paragraph shows that we can find an analytic continuation of our functions as long as z_1 lies in the upper half-plane, z_2 and z_3 in the lower half-plane, and as long as $\operatorname{Im}(r) > 0$. In the limiting case when r comes to the negative, real axis the circle representing the z_3 cut in Fig. 19 degenerates into a straight line that coincides with the left S-line. At the same time, the line representing the z_2-cut comes down to the real axis and coincides with the right S-line. These two cuts stop us from carrying the analytic completion further. According to (56 a) (cf. also (52)), a negative real value of r is a point on the analytic hypersurface F'_{23}. Using the complete symmetry of our domain we further conclude that, when z_k lies in one half-plane and z_l and z_m in the other, we can at least continue our function to that part of the analytic hypersurface F'_{lm} which lies in the same half-plane as z_l and z_m. We now finish by remarking that the F'_{kl} surfaces do not intersect each other and do intersect the cuts in the points P and in the origins. The corners formed at the points P are of the kind shown in Fig. 17 a and therefore do not allow us to continue our functions further. The corners formed at the origins are of the type shown in Fig. 17b, but are here the analytic manifolds $z_k = 0$. Therefore, it is not possible to continue through these corners either. Besides, the explicit example mentioned earlier from perturbation theory in p-space (cf. Appendix III) shows that we certainly cannot continue beyond the curves F'_{kl}.

Summarizing, we have proved: When the point z_k lies in one half-plane and the two points z_l and z_m in the other, the holomorphy envelope of our domain is given by that piece of the analytic hypersurface F'_{lm} that lies in the relevant region (cf. Fig. 18). This solves one part of our problem.

VIII. Analytic Completion through the Corners Formed by the F_{kl}-Curves.

When all points z_k lie in the same (upper) half-plane, the given domain is bounded by pieces of the hypersurfaces F_{kl} in (31) and (47). (Cf. Figs. 12 and 13). This domain is not a natural domain of analyticity because of the corners appearing at the intersections of these F_{kl}-surfaces. One such corner is shown in Fig. 13. A different intersection of a similar corner is obtained, e. g., in Fig. 12 for $y_1 = y_2$. In that case, the two surfaces F_{13} and F_{23} coincide in the picture and form a corner in the boundary. Here, the situation is more complicated than the situation discussed in the previous paragraph because the domain with singularities is not separated into several parts, each with only one corner in it. Rather, all three corners formed by the intersections $F_{12} \cap F_{23}$, $F_{12} \cap F_{13}$, and $F_{13} \cap F_{23}$ have to be considered together. Further, we have no guide from perturbation theory results in this case, as the function computed in Appendix III has no singularities when all the z_k's lie in the same half-plane. However, it can be shown by examples that our functions might have singularities also in this case[18]. Since our principle of introducing the parameter of the expected answer as a separate variable does not work until we have some conjecture as to what the answer might be, this appears to be a somewhat desperate situation. Nevertheless, as the transformation (56) worked so well in the previous paragraph, we optimistically try a similar transformation here. To make it more symmetric in all the variables we introduce a variable v instead of z_1 in (56) and write

$$\left.\begin{aligned} z_1 &= u + v, \\ z_2 &= u - r, \\ z_3 &= r\frac{v}{u}. \end{aligned}\right\} \tag{70}$$

[18] Such examples have been constructed by R. Jost and were explained to us by H. Lehmann. We want to thank both professors Jost and Lehmann for valuable discussions.

The inverse of this transformation is

$$\left.\begin{aligned} r &= \frac{1}{2}\left[z_1 - z_2 - z_3 + \sqrt{\lambda(z)}\right], \\ u &= \frac{1}{2}\left[z_1 + z_2 - z_3 + \sqrt{\lambda(z)}\right], \\ v &= \frac{1}{2}\left[z_1 - z_2 + z_3 - \sqrt{\lambda(z)}\right]. \end{aligned}\right\} \qquad (70\,\mathrm{a})$$

As is most clearly seen from (70a), real values of r, u, or v correspond to the analytic hypersurfaces F_{23}, F_{12}, and F_{13}, resp. As before[17], it does not matter which sign we choose for the root in (70a), if only we take the same root in all three equations.

In these new variables our boundary curves are given by

$$\left.\begin{aligned} &F_{12}\colon\ u = -\varrho \quad \text{or} \quad r = u + \varrho\frac{u}{u+v}, \\ &F_{23}\colon\ r = \varrho \quad\ \ \text{or} \quad r = u - \varrho\frac{u}{v}, \\ &F_{13}\colon\ v = -\varrho \quad \text{or} \quad r = \quad -\varrho\frac{u}{u+v}, \\ &\qquad z_1\text{-cut:}\ u + v = \varrho, \\ &\qquad z_2\text{-cut:}\ r = u - \varrho, \\ &\qquad z_3\text{-cut:}\ r = \varrho\frac{u}{v}, \\ &\qquad\quad 0 < \varrho < \infty. \end{aligned}\right\} \qquad (71)$$

If we fix the values of u and v and plot all these curves in the r-plane, they are all straight lines, except the z_1-cut and one branch each of F_{12} and F_{13}, which are not seen at all (unless $\mathrm{Im}\,(u+v) = 0$ or $\mathrm{Im}\,u = 0$ or $\mathrm{Im}\,v = 0$, in which case the whole r-plane lies in one of these manifolds). In this way, we might make a new set of pictures and replace the complicated curves in Figs. 11–13 by straight lines. However, the variables u, v, and r are somewhat less useful for physical applications and we do not want to use the u, v, r picture otherwise than as a tool for calculations. Instead of doing all this plotting, we go one step further at once and introduce the following transformation:

$$\left.\begin{aligned} u &= u^{(0)}, \\ v &= v^{(0)}(1+t), \\ r &= \frac{r^{(0)}}{1+t}. \end{aligned}\right\} \qquad (72)$$

This transformation has the property that the point $t=0$ corresponds to an arbitrarily chosen point $u^{(0)}$, $v^{(0)}$, $r^{(0)}$. Further, all points t correspond to the same value for u and for the product vr. The curve $v=-\varrho$ is mapped on the line $t=-1-\varrho/v^{(0)}$ and the curve $r=\varrho$ on the line $t=-1+r^{(0)}/\varrho$. The other branch of F_{23} in (71) is also mapped on a straight line, viz. $t=-1+r^{(0)}/u^{(0)}+\varrho/v^{(0)}$, while the other branch of F_{13} is repre-

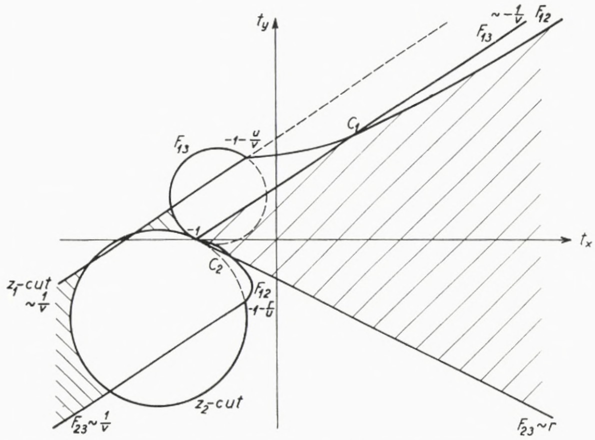

Fig. 22. The t-plane after the transformation (72). For simplicity, we have written u instead of $u^{(0)}$ etc. in this picture.

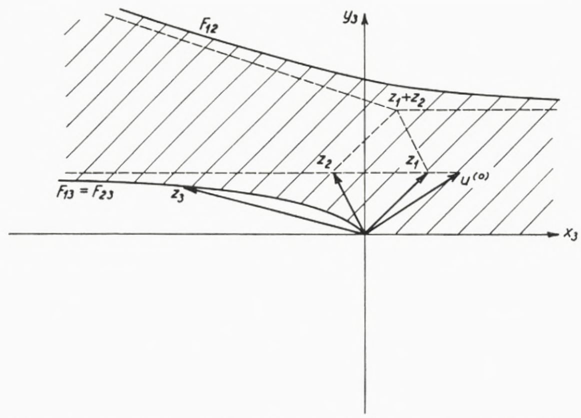

Fig. 23. Typical configuration of the z_k when $r^{(0)}$ and $-v^{(0)}$ are positive and real.

sented by the circle $t=-1-r^{(0)}\,u^{(0)}/\,(\varrho\, u^{(0)}+r^{(0)}\,v^{(0)})$. Further, the z_1-cut is given by the straight line $t=-(u^{(0)}+v^{(0)})/v^{(0)}+\varrho/v^{(0)}$ and the z_2-cut by the circle $t=-1+r^{(0)}/(u^{(0)}-\varrho)$. These curves are shown in Fig. 22. The z_3-cut and one branch of F_{12} is not seen in the t-plane. The other branch of F_{12} is a rather complicated curve given by

$$\frac{r^{(0)}}{1+t}=u^{(0)}+\varrho\,\frac{u^{(0)}}{u^{(0)}+v^{(0)}(1+t)}. \tag{73}$$

It has two disconnected parts, one of which starts from the point $-1-u^{(0)}/v^{(0)}$ and approaches an asymptote in the $-1/v^{(0)}$ direction. The other part goes from the point -1 to the point $-1+r^{(0)}/u^{(0)}$. Fortunately, the details of this curve are not important for the following discussion.

The configuration shown in Fig. 22 corresponds to $u^{(0)}$ and $v^{(0)}$ in the upper half-plane, while $r^{(0)}$ is in the lower half-plane. In the limiting case when $r^{(0)}$ and $-v^{(0)}$ are both positive and real, we are according to (71) on the corner between F_{23} and F_{13} for $t=0$. In z_k-space we then have the situation indicated in Fig. 23. If now $r^{(0)}$ is moved into the upper half-plane or $-1/v^{(0)}$ into the upper half-plane (or both), one sees from (70) that the point z_3, and therefore also the origin in the t-plane, lie inside the domain of analyticity[19]. However, if we move both $r^{(0)}$ into the lower half-plane

[19] If we change, say, only $r^{(0)}$ in the way just indicated and keep $v^{(0)}$ fixed, the point z_3 still lies on the curve F_{13}. However, for this change it follows from (70) that z_2 moves downwards, and therefore F_{13} lies below F_{23}, and the point z_3 lies inside the domain of analyticity (cf. Fig. 12). In a similar way, one sees that z_3 lies on F_{23} and therefore inside the domain of analyticity if only $v^{(0)}$ is moved into the upper half-

and $v^{(0)}$ into the upper half-plane, it can also be seen from (70) that the origin in the t-plane lies outside the domain of analyticity[19], provided that the particular part of the corner we happen to be looking at is on the boundary of our domain at all. Whether or not that is the case is a question of the relative magnitudes of $-r^{(0)}v^{(0)}$ and $|u^{(0)}|^2$. In Fig. 22, this is reflected in the position of the curve F_{12} relative to the origin. If we have the position shown in Fig. 22, where no branch of F_{12} encircles the origin, the origin is outside the domain of analyticity. The excluded domain is then bounded by the two straight lines F_{13} and F_{23} and by pieces of the curve F_{12}. We now simplify our problem a bit by considering as excluded also those pieces that lie between the two straight lines and the F_{12} curve. This means that we make our originally given domain of analyticity somewhat smaller than it really is. It still has the corner between F_{13} and F_{23}, but not the other two corners shown in C_1 and C_2 of Fig. 22. The holomorphy envelope that we compute for this simplified domain always lies outside the holomorphy envelope of the complete domain. In this simplified domain, the corner between F_{12} and F_{13} is the only corner we have and is always relevant.

We now have a situation somewhat analogous to Fig. 19 with an excluded wedge which shrinks down to a line and disappears when we pass the corner. As before, the excluded domain is open at infinity and attached to a fixed point (-1) at the edge, so it is impossible to introduce a closed path of integration around it. As before, we can improve upon this situation by introducing infinitesimal "curvature terms" in analogy with Eq. (58). Here, it is sufficient to write, instead of (72),

$$\left.\begin{aligned} v &= v^{(0)}\left[1+t+\varepsilon\, e^{-i\delta}\, v^{(0)} t^2\right], \\ r &= r^{(0)}\left[1+t-\frac{\varepsilon'}{r^{(0)}}\, e^{i\delta'}\, t^2\right]^{-1}, \end{aligned}\right\} \qquad (74)$$

with ε, ε', δ, and δ' real infinitesimal numbers. The two straight lines in Fig. 22 are now changed to two hyperbolas with centers at the points $\frac{-1}{2\,\varepsilon v^{(0)}}\, e^{i\delta}$ and $\frac{r^{(0)}}{2\,\varepsilon'}\, e^{-i\delta'}$, resp. The two curves start from the points $\frac{e^{i\delta}}{2\,\varepsilon v^{(0)}}\left[-1+\sqrt{1-4\,\varepsilon e^{-i\delta}\, v^{(0)}}\right] \approx -1-\varepsilon v^{(0)}\, e^{-i\delta}$ and $\frac{r^{(0)}}{2\,\varepsilon'}\, e^{-i\delta'}\left[1-\sqrt{1+\frac{4\,\varepsilon'}{r^{(0)}}\, e^{i\delta'}}\right] \approx -1+\frac{\varepsilon'}{r^{(0)}}\, e^{i\delta'}$. The slopes of the asymptotes are given by $\frac{1}{v^{(0)}}\, e^{i\frac{\delta}{2}}$ and $\frac{i}{v^{(0)}}\, e^{i\frac{\delta}{2}}$ for the first hyperbola, and by $-r^{(0)}\, e^{-i\frac{\delta'}{2}}$ and $-ir^{(0)}\, e^{-i\frac{\delta'}{2}}$ for the second one. If the two angles δ and δ' fulfil the inequalities $\sin\delta > 4\,\varepsilon v_y^{(0)}$ and $\sin\delta' > 4\,\varepsilon'\left(\frac{1}{r^{(0)}}\right)_y$, we get the situation indicated in Fig. 24. (If the two inequalities

plane. If we then move first one point and then the other, the first moving in the direction just indicated, we see from Fig. 22 that no piece of the boundary crosses the origin. Therefore, the origin is still inside the domain of analyticity. Finally, if we move $r^{(0)}$ into the lower half-plane, z_3 lies on F_{13}, which, however, is now the boundary as the point z_2 now lies above z_1. Moving $-1/v^{(0)}$ into the upper half-plane, we then get the point z_3 outside the domain of analyticity.

for δ and δ' are not fulfilled, the hyperbolas do not intersect at infinity, but approach their asymptotes from the other side). With the aid of the method used before, one can show that the intersections between the two hyperbolas correspond to points on the corner between F_{13} and F_{23} and that only the region between these two intersections lies outside the domain of analyticity (cf. Fig. 24).

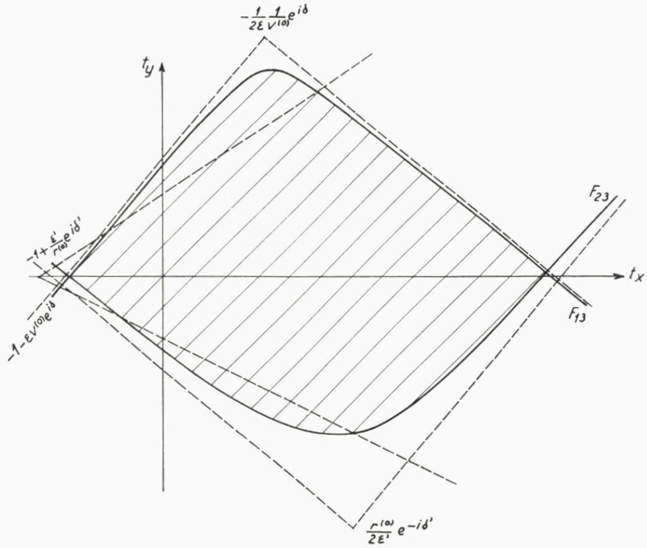

Fig. 24. The t-plane after the transformation (74).

As long as the angle between the directions $r^{(0)}$ and $-1/v^{(0)}$ is smaller than π, the region lying outside the domain of analyticity is finite, and we can introduce a closed path of integration around it and use the integral (55). For those positions of $r^{(0)}$ and $v^{(0)}$ which we are discussing now, it follows from Fig. 23 that z_1 and z_2 are always in the upper half-plane. For the corner to be on the boundary it is then necessary that also $z_3 = rv/u$ is in the upper half-plane. As $u^{(0)}$ lies in the upper half-plane, it follows from this condition that the angle between $r^{(0)}$ and $-1/v^{(0)}$ is smaller than π. In all cases of interest we can then continue our function analytically into the shaded domain in Fig. 22 – at least as long as $u^{(0)}$ lies in the upper half-plane. When $u^{(0)}$ becomes real and positive, the endpoints of the two circles in Fig. 22 lie on the two lines F_{13} and F_{23} and it is not possible to continue our function further using this transformation. According to (71), positive real values of u correspond to the manifold F'_{12}, and we have the following result: Our functions can at least be continued analytically from the corner between F_{13} and F_{23} to the analytic hypersurface F'_{12}. Using the symmetry of our domain we can generalize this statement to other permutations of the indices of the F_{kl} surfaces. Instead of Fig. 12 we then get Fig. 25. In the case shown in Fig. 13, the curve F'_{12} already lies inside the domain of analyticity and there is nothing to stop our analytic completion as long as all z_k's are in the same half-plane. Therefore, the whole cut plane lies inside the domain of analyticity in that case.

The domain shown in Fig. 25 is not a natural domain of analyticity, as it has a corner at the intersection between the two analytic hypersurfaces F'_{12} and F'_{23}. There are similar corners at the other intersections of the F'_{kl}-surfaces which are not shown in our figure. To get further, it is informative to plot the corner shown in Fig. 25 in the plane of the variable r in (70) for fixed values of u and v. We then get, e. g.,

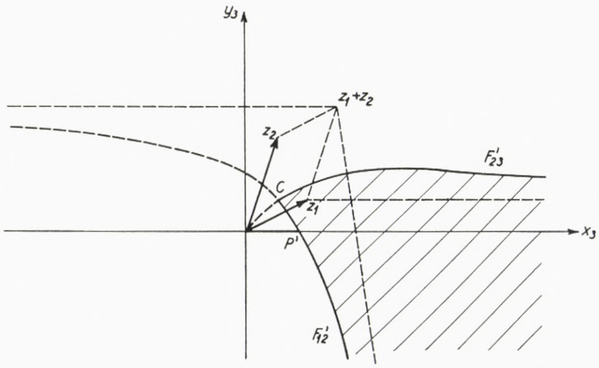

Fig. 25. The analyticity domain after the transformation (74).

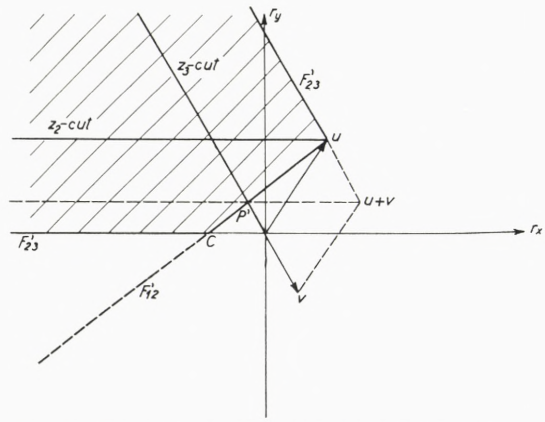

Fig. 26. The corner in Fig. 25 shown in the r-plane for fixed values of u and v.

a figure like Fig. 26. We have here put u in the upper half-plane and v in the lower one in such a position that $z_1 = u + v$ lies in the upper half-plane. We are then mainly interested in the region where z_2 and z_3 also lie in the upper half-plane. i. e., the region below the line $r = u - \varrho$ and to the left of the line $r = \varrho\, u/v$ (cf. Eqs. (71)). For the other parts of the diagram we have proved in the previous paragraph that one of the curves F'_{kl} is the holomorphy envelope for our domain. For completeness, the relevant F'_{kl}-curves have also been plotted in Fig. 26. We now make the important observation that the convex closure of the domain in the left part of Fig. 26, i. e., the horizontal line through the point P' (the intersection between the z_3-cut and F'_{12}) is an analytic hypersurface. Its equation is given by $r = u + v - \varrho$. As holomorphy envelopes are sometimes connected with convex closures[20], this surface is a possible conjecture for our answer. To follow up this idea we introduce the parameter of this surface as a new variable q and make the transformation

$$\left.\begin{aligned} q &= u + v - r\,, \\ w &= (u+v)\left(1 - \frac{r}{u}\right), \\ s &= -r\,. \end{aligned}\right\} \qquad (75)$$

[20] E. g., the holomorphy envelope to a "tube" is the convex closure of the tube. Cf. Bochner and Martin in ref. 12.

The variable w is only introduced to make the boundary curves into relatively simple geometrical objects also in the new q, w, s-space and is of no deeper significance. Our boundary curves now become

$$\left.\begin{aligned} &F'_{12}:\ w=\varrho \ \text{ or } \ w=q-s+\frac{s(q-s)}{\varrho}, \\ &F'_{23}:\ \ s=\varrho \ \text{ or } \ w^2+w(\varrho-q)+\varrho(s-q)=0, \\ &F'_{13}:\ w=q-s-\varrho \ \text{ or } \ w=q-s+\frac{s(q-s)}{q-s-\varrho}, \\ &\qquad z_1\text{-cut}:\ s-q+\varrho=0, \\ &\qquad z_2\text{-cut}:\ w=q-s+\frac{s(q-s)}{\varrho-s}, \\ &\qquad z_3\text{-cut}:\ w=q-\varrho. \end{aligned}\right\} \quad (76)$$

These curves are plotted in the w-plane for fixed values of q and s in Fig. 27. We have chosen q in the first quadrant and s in the fourth. Therefore, $z_1=q-s$ lies in the upper half-plane. We are then interested in that part of the w-plane that lies below the horizontal line through q and inside the circle representing the z_2-cut. With the technique used above on several occasions, one can prove that the intersections between the circle representing F'_{13} and the real axis correspond to the corners we are interested in. If the F'_{13} circle does not intersect the real axis at all (i. e. when the two points q and $q-s$ lie sufficiently close to each other), the whole interesting region of the w-plane lies inside the domain of analyticity. When the

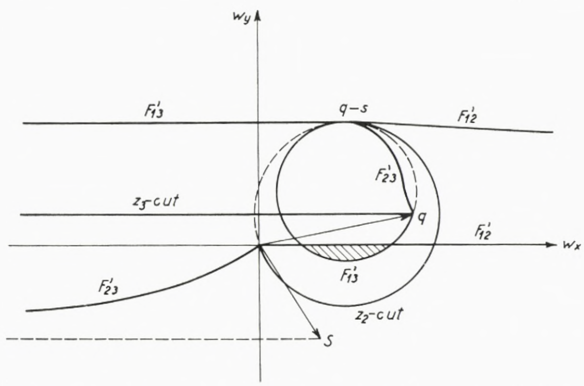

Fig. 27. The w-plane after the transformation (76).

circle dips below the real axis, the shaded region in Fig. 27 lies outside the domain of analyticity. In this case, this shaded region is already entirely separated from all other singularities of the function and we can immediately put a path of integration around it and apply the technique with the integral (55) without any use of special curvature terms. In this way, we can continue our functions analytically over the whole w-plane as long as the F'_{13}-circle lies inside the circle representing the z_2-cut. These two circles coincide when q becomes real and positive, i.e., our new boundary is the analytic hypersurface $q = \varrho$. In terms of the variables z_k, this surface is given by

$$\varrho^2 - \varrho(z_1 + z_2 + z_3) + z_1 z_2 + z_2 z_3 + z_1 z_3 = 0. \tag{77}$$

This surface is entirely symmetric in all the z_k's. This means that, if we do not start our analytic completion from the corner between the F'_{12} and F'_{13} but from some

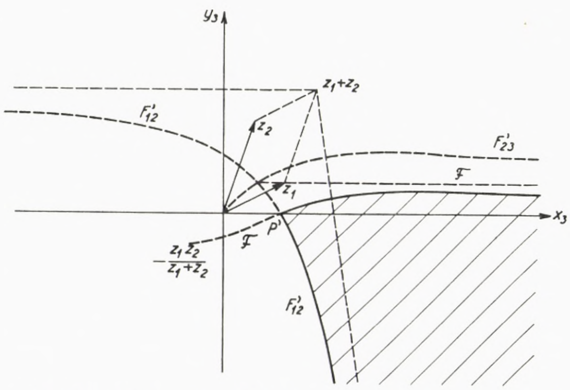

Fig. 28. The curve (77) ($\mathfrak{F}$) for $y_1 > 0$ and $y_2 > 0$ and $x_1 y_2 + x_2 y_1 > 0$.

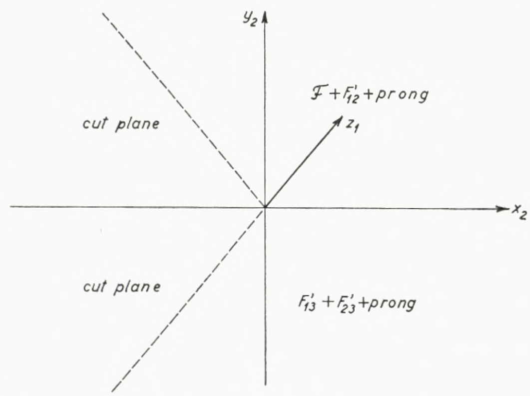

Fig. 29. General survey of the holomorphy envelope of our domain. The figure indicates the position of z_2 relative to z_1 for the different surfaces to be relevant. $\mathfrak{F}$ is the surface in Eq. (77).

of the other corners, we reach the same boundary (77) in all cases. When all the points z_1, z_2, and z_3 lie in one half-plane, one can further show that this new surface only intersects the curve F'_{kl} on the real axis of the variable z_m[21]. Therefore, this surface defines a natural domain of analyticity and is the holomorphy envelope of our given domain when all the points z_k lie in the same half-plane. The general character of this surface is shown in Fig. 28, while Fig. 29 gives a general survey of the holomorphy envelope for our domain, analogous to the survey of the domain itself given

[21] This is seen directly if we eliminate the parameters ϱ in (77) and in (31) (or (46)) and write the surface as an equation between x_k and y_k. In this way, we find that all surfaces can be written in the form

$$\begin{aligned}
\lambda(x) + \lambda(y) + 8\Delta &= \sqrt{[\lambda(x) - \lambda(y)]^2 + 4\mu^2(x,y)},\\
\lambda(x) &= x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3,\\
\lambda(y) &= y_1^2 + y_2^2 + y_3^2 - 2y_1y_2 - 2y_1y_3 - 2y_2y_3,\\
\mu(x,y) &= x_1(y_1 - y_2 - y_3) + x_2(y_2 - y_3 - y_1) + x_3(y_3 - y_1 - y_2) = \frac{1}{2}\,\mathrm{Im}\,\lambda(z),\\
\Delta &= y_k y_l \text{ for } F'_{kl},\\
\Delta &= y_1y_2 + y_1y_3 + y_2y_3 \text{ for } (77).
\end{aligned}$$

For two of these surfaces to intersect, we must have the corresponding Δ equal. Therefore, F'_{kl} does not intersect (77), unless $y_m(y_k + y_l) = 0$. As all y_k's have the same sign this is impossible, unless one or more of them are zero.

in Fig. 14. The curve (77) has the positive, real axis as asymptote and intersects its asymptote in the same point P' as the curve F'_{12}. For completeness, we also want to mention that when z_1 and z_2 both approach the positive, real axis, the curves F'_{12} and (77) in Fig. 28 and the curves F'_{23} and F'_{13} in Fig. 18 become mushroom-like in structure and no finite point in the z_3 plane lies inside the domain of analyticity.

IX. Discussion.

Our investigation has led to a domain of analyticity bounded by the following seven analytic hypersurfaces.

$$\left.\begin{array}{l} \text{Cuts:}\ \ z_k = \varrho \geq 0,\ k = 1, 2, 3, \\ F'_{kl}: z_m = z_k + z_l - \varrho - \dfrac{z_k\, z_l}{\varrho};\ 0 < \varrho < \infty;\ y_m\, y_k \leq 0;\ y_m\, y_l \leq 0, \\ \mathfrak{F}:\ \ z_1 z_2 + z_2 z_3 + z_1 z_3 - \varrho\,(z_1 + z_2 + z_3) + \varrho^2 = 0;\ 0 < \varrho < \infty;\ y_1\, y_2 \geq 0;\ y_1\, y_3 \geq 0\,. \end{array}\right\} \qquad (78)$$

It is the largest domain in which *every* function F^{ABC} and H^A satisfying the requirements enumerated in the Introduction is analytic. The imposition of any additional physical requirement will further restrict the class of permissible F^{ABC} and H^A and may give rise to an enlarged domain of analyticity. For example, it is easy to see that, when there are non-zero lower limits on the mass spectrum of the theory, the analyticity domain of the function H^A is always larger than that given by (78). On the other hand, imposing the boundedness restriction on the functions F^{ABC}, which was referred to in footnote 9 (and which we have ignored in our previous computation of the holomorphy envelope), does not increase the size of the domain of analyticity. One sees this by constructing functions analytic in the domain, having the boundedness property and singularities on the boundary. The calculations of Appendix III provide such examples for the F'_{kl} boundary and the cuts, and it is not difficult to construct others with singularities on $\mathfrak{F}$.

To make the results we have presented here useful for practical applications, it would also be very desirable to have some kind of a representation for the most general function analytic in our domain, but with singularities in arbitrary points outside. There is one technique available for such a purpose, viz., an application of the Bergmann-Weil[22] integral formula. This formula, which can be derived by a repeated application of Stoke's theorem in our six-dimensional space, expresses our analytic function for arbitrary values of the points z_k *inside* the domain of analyticity in terms of the boundary values of the function on a certain subset of the boundary. This subset, the so-called "distinguished boundary", consists of the three-dimensional

[22] An account of the derivation of this formula and its relation to other representation formulae is contained in F. SOMMER, Math. Ann. **125**, 172 (1952).

"intersections of the intersections" of the five-dimensional corners. This manifold can be computed for our case and turns out to consist of points of the form

$$z_1 = -(x-x')^2,\ z_2 = -(x'-x'')^2,\ z_3 = -(x-x'')^2$$

with x, x', x'' real vectors at least two of which are time-like or light-like. These form a subset of those points that are obtained in the limits indicated in (19) and (21). Therefore, it is possible to generate the whole analytic function with the aid of an integral involving (a certain rather complicated kernel and otherwise) nothing but its values at certain "physical points". Unfortunately, this representation has not shown itself to be very useful so far, and we do not want to give it here. We hope that further investigations along these lines will yield results of somewhat greater practical value.

Acknowledgement.

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Appendix I.

The derivation of the explicit boundary curves F_{12} and S, given in the main text in connection with Eqs. (31) and (40), has several advantages. The argument is absolutely rigorous, as its makes use of only the geometrical definition (18) of the domain and also involves only elementary, algebraic manipulations. On the other hand, it appears somewhat miraculous that the answer turns out to be an analytic hypersurface, a fact which is very important for the following discussion. Further, some skill is undeniably required in arranging the algebraic manipulations in such a way that they lead to the desired answer. In this Appendix, we shall give another derivation of the same boundary. This derivation makes use of formal integral representations of the functions involved. To make such an argument rigorous, one has to justify the inversions of orders of integrations etc. in some detail – a task we do not want to undertake. Without insisting on the epsilontic points, we present the alternative derivation here, because we feel that it might give some additional insight into the problem. Besides, this new derivation immediately yields the answer in the form of an analytic hypersurface. It might even be argued that it would have been practically impossible to find the systematic approach in the main text without having found the answer beforehand in some other way.

If we once and for all decide to invert orders of integrations freely, we may write, e. g., the function $F^{ABC}\,(x-x',\, x'-x'')$ in the following way:

$$\left.\begin{aligned} F^{ABC}(x-x', x'-x'') &= \frac{1}{(2\pi)^6}\iint dp\, dp'\, e^{ip(x-x')+ip'(x'-x'')}\,\Theta(p)\,\Theta(p')\,G^{ABC}(-p^2, -p'^2, -pp') \\ &= \int_0^\infty da \int_0^\infty db \int_{\sqrt{ab}}^\infty dc\, G^{ABC}(a,b,c)\,\Delta_3^{(+)}(x-x', x'-x'', a, b, c), \end{aligned}\right\} \quad (A.1)$$

$$\Delta_3^{(+)}(x, y, a, b, c) = \frac{1}{(2\pi)^6}\iint dp\, dp'\, e^{ipx+ip'y}\,\delta(p^2+a)\,\delta(p'^2+b)\,\delta(pp'+c)\,\Theta(p)\,\Theta(p'). \quad (A.2)$$

The integrations over p and q' in (A. 2) can be done with the aid of standard methods. The result is[23]

$$\left.\begin{aligned} \Delta_3^{(+)}(x, y, a, b, c) &= \frac{1}{(2\pi)^4}\sqrt{c^2-ab}\,\Theta\left(c-\sqrt{ab}\right)\int_0^\infty \frac{d\alpha\, J_0(\sqrt{\alpha})}{(\alpha-q)^2-r} \\ &= \frac{2i}{(4\pi)^3}\frac{\sqrt{c^2-ab}}{\sqrt{r}}\,\Theta\left(c-\sqrt{ab}\right)\left[H_0^{(1)}\left(\sqrt{q+\sqrt{r}}\right)-H_0^{(1)}\left(\sqrt{q-\sqrt{r}}\right)\right], \end{aligned}\right\} \quad (A.3)$$

$$q = -ax^2 - by^2 - 2cxy = z_1 a + z_2 b + (z_3 - z_1 - z_2)c, \quad (A.3)$$

$$r = 4\left[(xy)^2 - x^2y^2\right]\left[c^2 - ab\right] = \lambda(z)\left[c^2 - ab\right]. \quad (A.3b)$$

If the square roots inside the Hankel functions are defined to have a positive imaginary part, Eq. (A. 3) clearly exhibits $\Delta_3^{(+)}$ as the boundary value of an analytic function of the three complex variables $z_1 = -x^2$, $z_2 = -y^2$, and $z_3 = -(x+y)^2$, regular everywhere except on the manifold

$$q \pm \sqrt{r} = \varrho, \quad (A.4)$$

where ϱ is a positive, real number. In particular, the Hankel functions have a logarithmic singularity for $\varrho = 0$ and a cut for $\varrho \neq 0$. The function F^{ABC} in (A. 1) is also the boundary value of an analytic function which might have singularities on the manifold (A. 4). For most other points, the Hankel functions in (A. 3) are exponentially decreasing for large values of a, b, and c. If only the weight function G^{ABC} is, say, bounded at infinity by a polynomial in a, b, and c (this assumption also lies behind the argument of ref. 3 and in the discussion in the main text, cf. 9), the integral representation (A. 1) defines an analytic function F^{ABC}. The only exception appears if one of the two complex numbers $q \pm \sqrt{r}$ goes to infinity along a direction parallel to the positive, real axis for large values of a, b, and c. As we shall see later, this only happens when z_1 or z_2 is on the positive, real axis.

When the variables a, b, and c in (A. 1) vary over the domain

$$a > 0;\ b > 0;\ c > \sqrt{ab}, \quad (A.5)$$

[23] The result of a similar calculation has been published by A. S. Wightman and D. Hall, Phys. Rev. **99**, 674 (1955). Unfortunately, this paper contains some misprints. These are corrected in (A. 3) here.

while z_1, z_2, and z_3 are kept fixed, the two points $q \pm \sqrt{r}$ each cover a certain region in the complex plane. If none of these regions contains the origin or any part of the positive, real axis, or extends to infinity in a direction parallel to the positive, real axis, the function F^{ABC} in (A. 1) is analytic for that value of z_1, z_2, and z_3. To investigate what this means in terms of a domain of analyticity for F^{ABC} in the six-dimensional z-space, we consider the mapping

$$t = z_3 - z_1 - z_2 + \sqrt{\lambda(z)}\sqrt{1 - \alpha\beta} + z_1\alpha + z_2\beta. \tag{A.6}$$

The expression $\lambda(z)$ is defined in (A. 3b) (or in (52a)), while α, β, and t have the following meaning:

$$\left.\begin{aligned} &\left.\begin{aligned} \alpha &= \frac{a}{c} > 0, \\ \beta &= \frac{b}{c} > 0, \end{aligned}\right\} \alpha\beta < 1, \end{aligned}\right\} \tag{A.7a}$$

$$t = \frac{1}{c}\left(q + \sqrt{r}\right). \tag{A.7b}$$

When a, b, and c vary over the domain (A. 5), the point $q + \sqrt{r}$ covers a region that is obtained by multiplying by the real number c every point of the region covered by t when α and β vary over the domain (A. 7a). The domain (A. 7a) is bounded by the three curves $\alpha = 0$, $\beta = 0$, and $\alpha\beta = 1$. In the t-plane, the curve $\alpha = 0$ corresponds to the straight line

$$t = z_3 - z_1 - z_2 + \sqrt{\lambda(z)} + \beta z_2;\ 0 < \beta < \infty, \tag{A.8a}$$

the curve $\beta = 0$ corresponds to

$$t = z_3 - z_1 - z_2 + \sqrt{\lambda(z)} + \alpha z_1;\ 0 < \alpha < \infty, \tag{A.8b}$$

while $\alpha\beta = 1$ is mapped on the hyperbola

$$t = z_3 - z_1 - z_2 + \alpha z_1 + \frac{1}{\alpha} z_2;\ 0 < \alpha < \infty. \tag{A.8c}$$

However, the region in the t-plane in which we are interested is not necessarily bounded by these three curves. It might happen that the mapping (A. 6) is singular along a certain curve. This means that some region in the t-plane is covered twice when α and β vary in their domain (A. 7a). In that case, the region in which we are interested is bounded by the curves (A. 8) and by the envelope

$$\left. D = \begin{vmatrix} \dfrac{\partial t}{\partial \alpha} & \dfrac{\partial t}{\partial \beta} \\[2ex] \dfrac{\partial t^*}{\partial \alpha} & \dfrac{\partial t^*}{\partial \beta} \end{vmatrix} = 0. \right\} \tag{A.9}$$

To simplify the handling of this determinant we introduce two real numbers ϱ_1 and ϱ_2 defined by

$$\sqrt{\lambda(z)} = \varrho_1 z_1 + \varrho_2 z_2. \qquad (A.10)$$

With this notation, an elementary calculation yields

$$D = \frac{z_1 z_2^* - z_1^* z_2}{2\sqrt{1-\alpha\beta}}\left[2\sqrt{1-\alpha\beta} - \varrho_1\beta - \varrho_2\alpha\right] \qquad (A.11\,a)$$

or

$$2\sqrt{1-\alpha\beta} = \varrho_1\beta + \varrho_2\alpha. \qquad (A.11\,b)$$

(When z_1/z_2 is real, a direct calculation yields $\alpha = \beta\, z_2/z_1$. This can be considered as a limiting case of (A. 11 b) when both ϱ_i become very large and their ratio tends to $-z_2/z_1$. If this is considered a degenerate case of our formulae, the following discussion is completely general). If Eq. (A. 11 b) is substituted into (A. 6), the desired envelope results. To obtain a symmetric expression we introduce

$$\sqrt{1-\alpha\beta} = \sigma;\ 0 \leq \sigma \leq 1 \qquad (A.12)$$

as independent parameter and obtain

$$\alpha = \frac{1}{\varrho_2}\left[\sigma \pm \sqrt{\sigma^2(1+\varrho_1\varrho_2) - \varrho_1\varrho_2}\right], \qquad (A.13\,a)$$

$$\beta = \frac{1}{\varrho_1}\left[\sigma \mp \sqrt{\sigma^2(1+\varrho_1\varrho_2) - \varrho_1\varrho_2}\right], \qquad (A.13\,b)$$

$$t = z_3 - z_1 - z_2 + \sigma\frac{1+\varrho_1\varrho_2}{\varrho_1\varrho_2}(z_1\varrho_1 + z_2\varrho_2) \pm \frac{z_1\varrho_1 - z_2\varrho_2}{\varrho_1\varrho_2}\sqrt{\sigma^2(1+\varrho_1\varrho_2) - \varrho_1\varrho_2}. \qquad (A.13\,c)$$

The signs in front of the square roots in (A. 13) have to be chosen in such a way that α and β are both positive. This might mean that none, one, or both of the alternative signs are allowed. In addition to (A. 12), the value of σ is also restricted by the condition that α and β are real, i. e. that the expression $\sigma^2(1+\varrho_1\varrho_2) - \varrho_1\varrho_2$ is positive. Under these circumstances (A. 13 c) defines a piece of a conic section. If $1+\varrho_1\varrho_2$ is positive, this curve is a hyperbola, while it is an ellipse if $1+\varrho_1\varrho_2$ is negative. In either case, the curve (A. 13 c) is tangent to the line (A. 8 a) at the point

$$t = z_3 - z_1 - z_2 + \sqrt{\lambda(z)} + \frac{2 z_2}{\varrho_1} \quad (\text{for } \sigma = 1), \qquad (A.14\,a)$$

to the line (A. 8 b) at the point

$$t = z_3 - z_1 - z_2 + \sqrt{\lambda(z)} + \frac{2 z_1}{\varrho_2} \quad (\text{for } \sigma = 1), \qquad (A.14\,b)$$

and to the hyperbola (A. 8 c) at the point

$$t = z_3 - z_1 - z_2 + \sqrt{-\varrho_1 \varrho_2}\left(\frac{z_1}{|\varrho_2|} + \frac{z_2}{|\varrho_1|}\right) \text{ (for } \sigma = 0\text{)}. \tag{A.14c}$$

The two points (A. 14a) and (A. 14b) correspond to the same value of σ, but to different signs of the square roots in (A. 13). Eq. (A. 14a) is relevant for our discussion only if $\varrho_1 > 0$, Eq. (A. 14b) only if $\varrho_2 > 0$, and Eq. (A. 14c) only for $\varrho_1 \varrho_2 < 0$. In all cases, two of the three points (A. 14) are the endpoints of the curve (A. 13c).

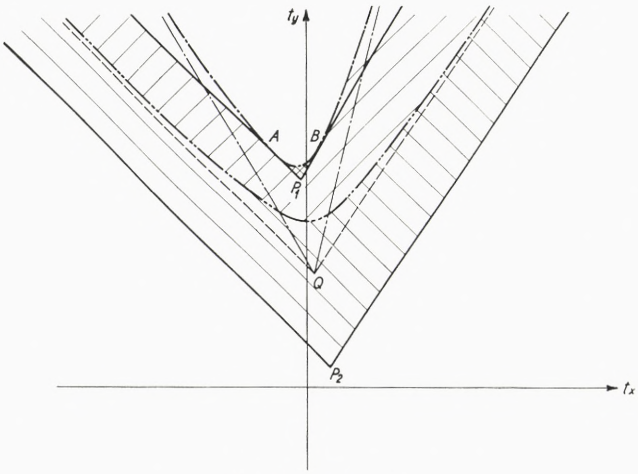

Fig. A.1 $\varrho_1 \varrho_2 > 1$

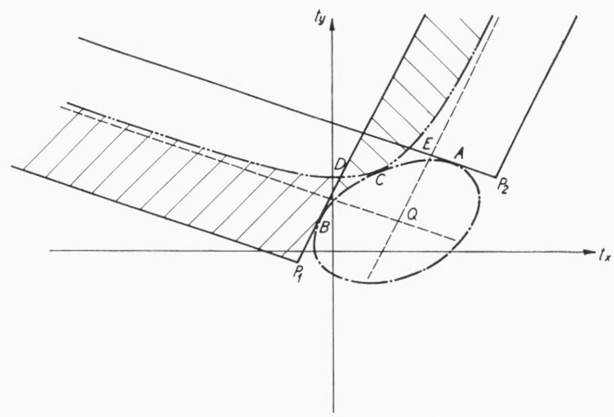

Fig. A.2 $\varrho_1 \varrho_2 < -1$

The Figs. A. 1 and A. 2 illustrate the mapping (A. 6). The curve drawn as $-\cdot\cdot-$ is the hyperbola (A. 8 c) and the curve drawn as $-\cdot-\cdot-$ is the conic section (A. 13 c). Q is the point $z_3 - z_1 - z_2$ and P_1 and P_2 are the points $z_3 - z_1 - z \pm \sqrt{\lambda(z)}$. A and A' are the points (A. 14 a), B is the point (A. 14 b) and C the point (A. 14 c).

Figures A. 1 and A. 2 illustrate two typical cases of our region in the t-plane. In the same figures we have also plotted the region that is obtained if the signs of ϱ_1 and ϱ_2 are *both* changed. We consider these cases together as they both appear simultaneously in (A. 4). In Fig. A. 1, the region ABP_1 is covered twice by the mapping with $\varrho_1 > 0$; $\varrho_2 > 0$. In Fig. A. 2, the region BDC is covered twice in one of the mappings and the region AEC twice in the other. If $0 < \varrho_1 \varrho_2 < 1$, the point P_1 lies below the curve (A. 8c) in Fig. A. 1. This does not change the qualitative character of the figure. For $\varrho_1 \varrho_2 = 0$, two of the parallel lines coincide. For $-1 < \varrho_1 \varrho_2 < 0$, the situation is similar to Fig. A. 2, but the curve (A. 13c) is still part of a hyperbola. For $\varrho_1 \varrho_2 = -1$, the points A and B coincide on the curve (A. 8c) and the curve (A. 13c) shrinks down to this point.

As the directions of the asymptotes of the curve (A. 8c) are determined by the points z_1 and z_2, Figs. A. 1 and A. 2 correspond to one of the z's in the first and the other in the second quadrant. In other cases, the whole figure changes in an obvious way.

If z_1 and z_2 are in opposite half-planes, the boundary of the domain of analyticity for F^{ABC} is obtained when the first piece of the region covered by our mapping (A. 6) passes through the origin. Instead of discussing the condition for the curves (A. 8) and (A. 13c) to go separately through the origin, we give at once a general discussion for any point of the mapping (A. 6). For $t=0$ we get

$$[z_3 - z_1(1-\alpha) - z_2(1-\beta)]^2 = (1-\alpha\beta)\,\lambda(z), \tag{A.15}$$

(A. 15) can be rearranged to read

$$\alpha + s + \frac{\beta}{4} s^2 = 0, \tag{A.16}$$

$$s = \frac{1}{z_1}\left[z_3 - z_1 - z_2 \pm \sqrt{\lambda(z)}\right]. \tag{A.16a}$$

The two equations (A. 16) (and (A. 16a)) are linear, complex relations between the two real numbers α and β. They therefore correspond to one real pair (α, β) each. For these two points in the α, β space, the Hankel functions in (A. 3) have a singular value. However, it turns out that the product $\alpha\beta$ in (A. 16) is given by

$$\alpha\beta = \frac{|s|^2}{[\mathrm{Re}\, s]^2} > 1. \tag{A.17}$$

The equality sign in (A. 17) holds only when the imaginary part of s is zero. As the value of the product $\alpha\beta$ in (A. 17) is, in general, outside the domain (A. 7), we find that the origin of the t-plane is normally not inside the region of our mapping. The exceptional case, which is therefore also the boundary of the analyticity domain of F^{ABC}, happens when the point z_1, z_2, z_3 lies on the analytic hypersurface

$$z_3 - z_1 - z_2 \pm \sqrt{\lambda(z)} + 2\,k z_1 = 0 \tag{A.18}$$

with k a real number. The corresponding values of α and β fulfil

$$\alpha - 2\,k + \beta k^2 = 0. \tag{A.19}$$

Therefore, k must not only be real but also positive. Eq. (A. 18) can be rewritten as

$$z_3 = z_1(1-k) + z_2\left(1 - \frac{1}{k}\right), \tag{A.20}$$

and is identical with the S-curve obtained earlier as the boundary when z_1 and z_2 lie in opposite half-planes.

The manifold (A. 20) was obtained as the condition for $t=0$ to be inside the region of our mapping, but without use of the condition that z_1 and z_2 lie in opposite half-planes. If they lie in the same half-plane, we can still conclude that the point $t=0$ is always outside our region, except on the manifold S. However, in this case S is not the boundary, as one of the points P_1 and P_2 (i. e. at least one of them) is always in the half-plane opposite to the half-plane of z_1 and z_2 on the manifold S. The real boundary of the domain of analyticity is then given by the condition for P_1 or P_2 to be on the positive, real, axis, or

$$z_3 - z_1 - z_2 \pm \sqrt{\lambda(z)} = 2r;\ 0 < r < \infty. \tag{A.21}$$

This can be rewritten as

$$z_3 = z_1 + z_2 + r + \frac{z_1 z_2}{r}, \tag{A.22}$$

and is identical with the manifold F_{12} earlier obtained as the boundary when z_1 and z_2 lie in the same half-plane.

Appendix II.

In this Appendix, we outline yet a third derivation of the boundary of $\mathfrak{M}$. The methods used are as rigorous as the derivation given in Section IV and are based on ref. 7. Apart from the results obtained there we use only elementary arguments; the proofs will be omitted[24].

Let $\mathfrak{T}_n$ (the tube in the notation of ref. 7) be the set of all points $\zeta_1, \ldots \zeta_n$ where $\zeta_j, j = 1, \ldots n$ are complex four-vectors of the form

$$\zeta_j = \xi_j - i\,\eta_j, \tag{A. 23}$$

and ξ_j and η_j are real four-vectors with

$$\eta_j^2 < 0,\ \eta_{j0} > 0. \tag{A.24}$$

Let $\mathfrak{T}_n'$ (the extended tube in the notation of ref. 7) be the set of all points $\Lambda\zeta_1, \ldots \Lambda\zeta_n$ where Λ is any element of the complex Lorentz group and $\zeta_1, \ldots \zeta_n \in \mathfrak{T}_n$. The image of $\mathfrak{T}_n$ (or $\mathfrak{T}_n'$) under the mapping

$$\zeta_1, \ldots \zeta_n \rightarrow \zeta_j \zeta_k,\ j, k = 1, \ldots n$$

is a subset, $\mathfrak{M}_n$, of all complex symmetric matrices of rank ≤ 4. (Evidently $\mathfrak{M}_2 = \mathfrak{M}$). Our problem is to give explicit formulae for the boundary of $\mathfrak{M}_n$ which we denote $\partial\mathfrak{M}_n$. We first study the boundary $\partial\mathfrak{T}_n'$, of $\mathfrak{T}_n'$ and show that it suffices to consider the subset B_n which is the common part $(\partial\mathfrak{T}_n) \cap (\partial\mathfrak{T}_n')$ of the boundaries of $\mathfrak{T}_n$ and $\mathfrak{T}_n'$.

[24] The second-named author has profited greatly from suggestions of R. Jost on the subject matter of this Appendix.

Lemma

Let the point $\zeta_1, \ldots \zeta_n$ belong to the boundary of the extended tube, $\partial \mathfrak{T}'_n$. Then,

a) if the matrix $\zeta_j \zeta_k$, $j, k = 1, \ldots n$ has rank 3 or 4, there exists a complex Lorentz transformation Λ such that the point

$$\Lambda\zeta_1, \ldots \Lambda\zeta_n \tag{A. 25}$$

lies in B_n;

b) if the matrix $\zeta_j \zeta_k$, $j, k = 1, \ldots n$ has rank 2 or 1, there exist a complex Lorentz transformation Λ, complex numbers α_j, $j = 1, \ldots n$, and a vector ω such that the point

$$\Lambda\zeta_1 + \alpha_1 \omega, \ldots . \Lambda\zeta_n + \alpha_n \omega \tag{A.26}$$

lies in B_n and $\omega^2 = 0 = \omega \Lambda \zeta_j$, $j = 1, \ldots n$;

c) if the matrix $\zeta_j \zeta_k$, $j, k = 1, \ldots n$ is of rank zero, i. e. $\zeta_j \zeta_k = 0$ for all j and k, there exist complex numbers α_j and β_j, $j = 1, \ldots . n$ and vectors ω_1 and ω_2 such that

$$\zeta_j = \alpha_j \omega_1 + \beta_j \omega_2 \quad \text{with} \quad \omega_1^2 = \omega_2^2 = \omega_1 \omega_2 = 0 . \tag{A.27}$$

The proof of this Lemma is based on Lemmas 2 and 3 of ref. 7. The points of B_n lie on the boundary of $\mathfrak{T}_n$ and are distinguished from points lying in the interior of $\mathfrak{T}'_n$ by the fact that no complex Lorentz transformation, Λ, carries them into $\mathfrak{T}_n$, i. e. if $\zeta_1, \ldots . \zeta_n \in B_n$, none of the points $\Lambda\zeta_1, \ldots \Lambda\zeta_n \in \mathfrak{T}_n$. We use this property of the points of B_n to partition it into disjoint subsets.

Definition

A point $\zeta_1, \ldots . \zeta_n \in \mathfrak{T}_n$ is in $B_n^{(j)}$ if

a) For every $j-1$ element subset $\zeta_{k_1}, \ldots . \zeta_{k_{j-1}}$ there exists a complex Lorentz transformation Λ such that $\Lambda\zeta_{k_1}, \ldots \Lambda\zeta_{k_{j-1}} \in \mathfrak{T}_{j-1}$.

b) There exists a j-element subset $\zeta_{l_1}, \ldots \zeta_{l_j}$ such that $\Lambda\zeta_{l_1}, \ldots \Lambda\zeta_{l_j}$ do not belong to $\mathfrak{T}_j$ for all complex Lorentz transformations Λ.

It turns out that, if there exists any Λ such that $\Lambda\zeta_{k_1}, \ldots \Lambda\zeta_{k_{j-1}} \in \mathfrak{T}_{j-1}$, there exists an infinitesimal Λ so that, if we write $\Lambda = 1 + M$, the determination of such a Λ is reduced to the problem of satisfying a set of inequalities of the form $\{Im\,[(1+M)\,\zeta_j]\}^2 < 0$, $\{Im\,[(1+M)\,\zeta_j]\}_0 > 0$. The left-hand sides of these inequalities are at worst second degree and first degree expressions, respectively, in the twelve real parameters of the Lorentz group. In general, it suffices to consider the linear terms and in that case one gets.

Lemma

A sufficient condition that $\zeta_1, \ldots \zeta_j \in \partial \mathfrak{T}_j$ be such that $\Lambda\zeta_1, \ldots \Lambda\zeta_j$ do not belong to $\mathfrak{T}_j$ for all complex Lorentz transformations Λ is that the inequalities

$$\sum_{k=1}^{j} M_{\mu\nu}\left(\xi_k^{\mu}\,\eta_k^{\nu}-\xi_k^{\nu}\,\eta_k^{\mu}\right)>0, \tag{A.28}$$

possess no non-trivial antisymmetric solution $M_{\mu\nu}$. (Here $\zeta_j = \xi_j - i\eta_j$). The condition is also necessary if none of the quantities $\xi_k^{\mu}\eta_k^{\nu} - \xi_k^{\nu}\eta_k^{\mu} = 0$ for all μ and ν.

The elementary theory of convex cones permits one to translate (A. 28) into the condition

$$\sum_{k=1}^{j} \lambda_k\left(\xi_k^{\mu}\,\eta_k^{\nu}-\xi_k^{\nu}\,\eta_k^{\mu}\right)=0 \tag{A.29}$$

for some non-negative λ_k, $k = 1, \ldots\, n$ such that $\sum_{k=1}^{j}\lambda_k > 0$. Equations of the form (A. 29) occur in the theory of exterior differential forms[25]. Suppose that among the η_k there are at most N linearly independent vectors. For notational convenience we suppose that $\eta_1, \ldots \eta_N$ have this property. Then

$$\eta_k = \sum_{l=1}^{N} \beta_{kl}\,\eta_l,\ k = N+1, \ldots n, \tag{A.30}$$

and the general solution of (A. 29) is given by

$$\lambda_j\,\xi_j + \sum_{k=N+1}^{n} \lambda_k\,\xi_k\,\beta_{kj} = \sum_{l=1}^{N} \alpha_{jl}\,\eta_l,\ j = 1, \ldots N,\ \alpha_{jl} = \alpha_{lj}. \tag{A.31}$$

We collect the previous information in a theorem.

Theorem

The set $B_n^{(j)}$ contains the following points $\zeta_1, \ldots \zeta_n$.

All points such that

1) $\eta_{k_1}, \ldots \eta_{k_j}$ lie on the light cone and the rest of the η's inside and

2) some subset $\eta_{k_1'}, \ldots \eta_{k_N'}$ of the $\eta_{k_1}, \ldots \eta_{k_j}$ form a maximal, linearly independent set

3) for some positive λ's

$$\lambda_{k_r'}\,\xi_{k_r'} + \sum_{s=N+1}^{j} \lambda_{k_s'}\,\xi_{k_s'}\,\beta_{k_s' k_r'} = \sum_{s=1}^{N} \alpha_{k_r' k_s'}\,\eta_{k_s'},\ r = 1, \ldots N, \tag{A.32}$$

where

$$\alpha_{k_r' k_s'} = \alpha_{k_s' k_r'};\quad r, s = 1, \ldots N$$

and $k_1', \ldots k_j'$ is a permutation of $k_1, \ldots k_j$, and

4) no subset of $\zeta_{k_1}, \ldots \zeta_{kj}$ belongs to $B_j^{(r)}$, $1 < r < j$.

[25] See, for example, E. Cartan, *Les Systèmes Différentiels Extérieurs et leurs Applications Géométriques*, Hermann 1945, p. 11. We are indebted to R. Deheuvels for this reference.

It turns out that all other points of $B_n^{(j)}$ are limiting cases of the points described in the theorem, but we will not go into such fine points here.

The preceding theorem parametrizes some of the points of $B_n^{(j)}$. For the purposes of the present paper, we need a parametrization of the corresponding scalar products in the cases $B_2^{(1)}$ and $B_2^{(2)}$. To close this Appendix, we carry out the required elementary calculations. The vectors of $B_2^{(1)}$ evidently correspond to $z_1 = -\zeta_1^2 \geq 0$ or $z_2 = -\zeta_2^2 \geq 0$ or both, and therefore yield the parts of the boundary we have previously called the cuts. For $B_2^{(2)}$ there are two cases to be considered, η_1 and η_2 linearly independent and η_1 and η_2 linearly dependent.

In the first case, we have

$$\xi_j = \sum_{k=1}^{2} \alpha_{jk}\,\eta_k; \quad \operatorname{sgn} \alpha_{jk} = \operatorname{sgn} \alpha_{kj}; \; j = 1, 2. \tag{A.33}$$

Therefore,

$$z_j = -\xi_j^2 + 2\,i\,\xi_j\eta_j = -2\,(\alpha_{jj}\,\alpha_{jk} - i\,\alpha_{jk})\,\eta_1\eta_2,\; j \neq k,\; j = 1, 2, \tag{A.34}$$

$$\left.\begin{aligned} z_3 &= z_1 + z_2 - 2\,(\xi_1\xi_2 - \eta_1\eta_2) + 2\,i\,(\xi_1\eta_2 + \xi_2\eta_1) \\ &= z_1 + z_2 - 2\,[(\alpha_{12}\,\alpha_{21} + \alpha_{11}\,\alpha_{22} - 1) + i\,(\alpha_{11} + \alpha_{22})]\,\eta_1\eta_2. \end{aligned}\right\} \tag{A.35}$$

Now, defining the positive number r by the equation

$$r = -2\,\alpha_{12}\,\alpha_{21}\,\eta_1\eta_2, \tag{A.36}$$

we have

$$z_3 = z_1 + z_2 + r + \frac{z_1\,z_2}{r}, \quad 0 < r < \infty, \tag{A. 37}$$

which coincides with (44). From (A. 34) it is evident that the boundary points obtained in this way have $y_1\,y_2 \geq 0$. A short examination of (A. 34) and (A. 35) makes clear that for any fixed z_1 aud z_2 satisfying $y_1\,y_2 > 0$, every point of (A. 37) is obtained from (A. 33).

In the second case, η_1 and η_2 are linearly dependent and, according to (A. 32), we have

$$\eta_2 = \beta_{21}\,\eta_1,\; \beta_{21} > 0;\; \lambda_1\,\xi_1 + \lambda_2\,\beta_{21}\,\xi_2 = \alpha_{11}\,\eta_1;\; \lambda_1 > 0,\; \lambda_2 > 0. \tag{A.38}$$

Consequently, the scalar products are

$$z_1 = -\xi_1^2 + 2\,i\,\xi_1\,\eta_1 = -\left(\frac{\lambda_2}{\lambda_1}\right)^2 \beta_{21}^2\,\xi_2^2 + 2\,\frac{\lambda_2}{\lambda_1}\frac{\alpha_{11}}{\lambda_1}\,\xi_2\,\eta_2 + 2\,i\,\frac{\lambda_2}{\lambda_1}\,\xi_2\,\eta_2, \tag{A.39}$$

$$z_2 = -\xi_2^2 + 2\,i\,\xi_2\,\eta_2, \tag{A.40}$$

$$z_3 = z_1 + z_2 - 2\left(-\frac{\lambda_2}{\lambda_1}\beta_{21}\,\xi_2^2 + \frac{\alpha_{11}}{\lambda_1\,\beta_{21}}\,\xi_2\,\eta_2\right) + i\left(-\frac{\lambda_2}{\lambda_1}\,\beta_{21}\,\xi_2\,\eta_2 + \frac{1}{\beta_{21}}\,\xi_2\,\eta_2\right). \tag{A.41}$$

If we introduce the notation

$$k = \frac{\lambda_1}{\lambda_2 \beta_{21}},$$

(A. 41) may be written

$$z_3 = z_1(1-k) + z_2\left(1 - \frac{1}{k}\right), \quad 0 < k < \infty, \tag{A.42}$$

which coincides with (45). It is evident from (A. 39) and (A. 40) that the boundary points obtained in this way have $y_1 y_2 \leq 0$. Again it is easy to see that, for any fixed z_1 and z_2 satisfying $y_1 y_2 < 0$, every point of (A. 42) is obtained from vectors of the form (A. 38).

Appendix III.

As an illustration of the analytic properties of the functions $F(z)$ and $H(z)$ discussed above we collect some results for these functions obtained from perturbation theory. As a first example, let us consider three different free fields $\Phi_i^{(0)}(x)$ with three different masses $m_i^2 = a_i$. We then put $A(x) = \Phi_1^{(0)}(x)\,\Phi_2^{(0)}(x)$; $B(x) = \Phi_2^{(0)}(x) \times \Phi_3^{(0)}(x)$ and $C(x) = \Phi_3^{(0)}(x)\,\Phi_1^{(0)}(x)$ in (17) and obtain[26]

$$\left.\begin{aligned} F^{ABC}(x-x', x'-x'') &= \langle 0 \,|\, \Phi_1^{(0)}(x)\,\Phi_2^{(0)}(x)\,\Phi_1^{(0)}(x')\,\Phi_3^{(0)}(x')\,\Phi_3^{(0)}(x'')\,\Phi_1^{(0)}(x'') \,|\, 0\rangle \\ &= -i\Delta^{(+)}(x-x', a_2)\,\Delta^{(+)}(x-x'', a_1)\,\Delta^{(+)}(x'-x'', a_3), \end{aligned}\right\} \tag{A.43}$$

where

$$\Delta^{(+)}(x-y, a_i) = -i\langle 0 \,|\, \Phi_i^{(0)}(x)\,\Phi_i^{(0)}(y) \,|\, 0\rangle = \frac{-i}{(2\pi)^3}\int dp\, e^{ip(x-y)}\,\delta(p^2 + a_i)\,\Theta(p). \tag{A.43a}$$

As is well known, the function $\Delta^{(+)}(x, a)$ in (A. 43a) is the boundary value of the Hankel function $\frac{-a}{8\pi} H_1^{(0)}(\sqrt{az})/\sqrt{az}$ with $z = -x^2$. This Hankel function is analytic in the whole cut z-plane and the function F^{ABC} in (A. 43) is the boundary value of an analytic function regular when all the z_k's vary independently in their cut planes. The domain of analyticity of this example is therefore much larger than the holomorphy envelope of the domain U. By computing the retarded commutator (22) and its Fourier transform we get a somewhat more interesting result. After straightforward calculations, the following expression for the analytic function $H^A(z)$ is obtained:

$$H^A(z_1, z_2, z_3) = \frac{1}{16\pi^2}\int\!\!\int\!\!\int_0^1 \frac{d\alpha_1\, d\alpha_2\, d\alpha_3\, \delta(1-\alpha_1-\alpha_2-\alpha_3)}{z_1\alpha_2\alpha_3 + z_2\alpha_3\alpha_1 + z_3\alpha_1\alpha_2 - a_1\alpha_1 - a_2\alpha_2 - a_3\alpha_3}. \tag{A.44}$$

[26] Expressions of this general character are obtained in perturbation theory when the field operators obey equations of motion whose right-hand sides, the "currents", are bilinear expressions in the fields. We may regard the expression (A. 43) as a typical case of the vacuum expectation value of the product of three current operators.

The variables α_i in (A. 44) are one-dimensional "Feynman variables" which have been introduced to simplify the formal integrations in p-space. The remaining three integrations in (A. 44) can be carried through, but lead to rather involved expressions. The result can be somewhat simplified if we differentiate (A. 44) with respect to the three masses a_i and add the results. In this way, we get

$$\left(\frac{\partial}{\partial a_1}+\frac{\partial}{\partial a_2}+\frac{\partial}{\partial a_3}\right) H^A(z_1, z_2, z_3) = \frac{1}{16\pi^2}\iiint_0^1 \frac{d\alpha_1\, d\alpha_2\, d\alpha_3\, \delta(1-\alpha_1-\alpha_2-\alpha_3)}{[z_1\alpha_2\alpha_3+\cdots-a_1\alpha_1-\cdots]^2}. \qquad \text{(A. 45)}$$

As we are interested only in the analytic properties of H as a function of the z's, (A. 45) is quite as useful for us as the original expression (A. 44). The three α-integrations in (A. 45) can be carried through and the result can be expressed in terms of elementary functions only. We write the result as

$$\sum_{k=1}^{3}\frac{\partial}{\partial a_k} H(z) = \frac{-1}{32\pi^2}\cdot\frac{1}{\Phi}\sum_{k=1}^{3}\frac{P_k}{\sqrt{R_k}}\log\frac{z_k-a_l-a_m+\sqrt{R_k}}{z_k-a_l-a_m-\sqrt{R_k}}, \qquad \text{(A. 46)}$$

$$\Phi = \sum_{k=1}^{3} z_k a_k^2 + \sum_{k<l} a_k a_l (z_m - z_k - z_l) + \sum a_k z_k (z_k - z_l - z_m) + z_1 z_2 z_3, \qquad \text{(A. 46 a)}$$

$$P_k = \frac{\partial\Phi}{\partial a_k}, \qquad \text{(A. 46 b)}$$

$$R_k = (z_k - a_l - a_m)^2 - 4\, a_l a_m \quad (k, l, m \text{ cycl.}). \qquad \text{(A. 46 c)}$$

We further note the identity

$$P_k^2 = \lambda(z) R_k + 4 z_k \Phi \qquad \text{(A. 46 d)}$$

between P_k, R_k, and Φ.

The function in (A. 46), and therefore also the function $H^A(z)$, has possible singularities at

(i) The points in which the functions

$$\frac{1}{\sqrt{R_k}}\log\frac{z_k-a_l-a_m+\sqrt{R_k}}{z_k-a_l-a_m-\sqrt{R_k}}$$

have singularities.

(ii) The points in which we have $\Phi = 0$.

To discuss case (i) we write the logarithms appearing in (A. 46) in the following way:

$$\frac{1}{\sqrt{R_k}}\log\frac{z_k-a_l-a_m+\sqrt{R_k}}{z_k-a_l-a_m-\sqrt{R_k}} = \int_0^1 \frac{d\alpha}{z_k\alpha(1-\alpha)-a_l\,\alpha-a_m(1-\alpha)}. \qquad \text{(A. 47)}$$

From this representation it is clear that the function in (A. 47) is analytic everywhere, except when $z_k\alpha(1-\alpha)-a_l\alpha-a_m(1-\alpha)$ vanishes for some value of α in the interval

(0,1). This only happens when z_k is real and positive and bigger than $(\sqrt{a_l}+\sqrt{a_m})^2$. This singularity has an immediate physical significance as it corresponds to the "threshold" for the creation of one real particle with mass $\sqrt{a_l}$ and another with mass $\sqrt{a_m}$. It may be remarked that the position of a singularity of this kind depends only on one of the variables z_k. It follows, e. g., from Fig. 29 that, if such a singularity is to occur in H^A, it must be on the positive, real axis. This is consistent with the result found here as long as the masses a_k are positive, real quantities.

In the case (ii) it follows from (A. 46d) that the squares of all the ratios $P_k/\sqrt{R_k}$ are equal (and equal to $\lambda(z)$). To handle the signs involved, we solve the equation $\Phi=0$ for z_3 and get

$$z_3 = a_1 + a_2 - \frac{1}{2\,a_3}\left[(z_1-a_2-a_3)(z_2-a_3-a_1) \pm \sqrt{R_1}\sqrt{R_2}\right]. \qquad \text{(A. 48 a)}$$

Substituting into R_3 in (A. 46c) we find

$$R_3 = \frac{1}{4\,a_3^2}\left[\sqrt{R_1}\,(z_2-a_1-a_3) \pm \sqrt{R_2}\,(z_1-a_2-a_3)\right]^2. \qquad \text{(A. 48 b)}$$

As the sign of $\sqrt{R_3}$ is of no consequence in (A. 46) we may choose

$$\sqrt{R_3} = \frac{1}{2\,a_3}\left[\sqrt{R_1}\,(z_2-a_1-a_3) \pm \sqrt{R_2}\,(z_1-a_2-a_3)\right] \qquad \text{(A. 48 c)}$$

and get

$$\frac{P_1}{\sqrt{R_1}} = \frac{P_3}{\sqrt{R_3}} = \pm\frac{P_2}{\sqrt{R_2}}. \qquad \text{(A. 48 d)}$$

The sign in the last expression in (A. 48d) is the same as the sign in front of the square roots in (A. 48a). We further find

$$z_3 - a_1 - a_2 + \sqrt{R_3} = \frac{-1}{2\,a_3}\left[z_1-a_2-a_3-\sqrt{R_1}\right]\left[z_2-a_1-a_3 \mp \sqrt{R_2}\right]. \qquad \text{(A. 49)}$$

With the aid of (A. 49) and (A. 48) we now get

$$\sum_{k=1}^{3} \frac{P_k}{\sqrt{R_k}} \log \frac{z_k-a_l-a_m+\sqrt{R_k}}{z_k-a_l-a_m-\sqrt{R_k}} = \frac{1}{\sqrt{\lambda(z)}}\log 1 = \frac{2\,n i\pi}{\sqrt{\lambda(z)}} \qquad \text{(A. 50)}$$

on the manifold $\Phi=0$. Therefore, the analytic function in (A. 46) has singularities in the case (ii) *only when the integer n in* (A. 50) *is different from zero.* The value of this integer has to be determined by a careful investigation of the branches of the logarithms in (A. 46). As is most easily seen from (A. 45), the function must be regular and real when all the z's lie on the negative, real axis. This requirement forces us to pick that branch of the logarithm that is real when the z's lie on the negative, real axis (note that the R_k's are positive there). Finally, we also note that it follows from the integral representation (A. 45) that the function H is analytic when the imaginary parts of z_k all have the same sign. In that case, the denominator never vanishes during the integration over the variables α_k.

As the function (A. 46) has been constructed in such a way that it can have no singularities in the domain of holomorphy U found in the main text, we may use the results there to simplify the discussion of the position of the singularities of (A. 46). If we fix z_1 and z_2, e. g., in the positions in Fig. 28, we have already proved that there can be no singularity above the real z_3-axis. Therefore, any point on the manifold $\Phi = 0$ with z_3 in the upper half-plane must have n in (A. 50) equal to zero. By putting $a_1 = a_2 = 0$ and choosing a suitable sign in front of the square roots in (A. 48a), we find that the curve F'_{12} lies in $\Phi = 0$. Therefore, $n = 0$ on the part of F'_{12} that lies above the real z_3-axis in Fig. 28. When z_3 passes through the point P' in Fig. 28, the logarithm involving z_3 in (A. 46) makes a jump while the two others remain fixed. Therefore, $n \neq 0$ on the lower part of F'_{12} in Fig. 28 and the function (A. 46) does have a singularity there. By choosing a_1 and a_2 different from zero, the curve given by (A. 48a) in Fig. 28 is displaced to the right of F'_{12}, and it follows in the same way that the function (A. 46) has a singularity in any point below the real axis and to the right of F'_{12} in Fig. 28 for suitable values of a_k. If we permute z_1 and z_3 or z_2 and z_3 and use the symmetry properties of (A. 46), it follows from this that our function has singularities inside the shaded domain in Fig. 18. The only other singularities of our function are of the kind (i) and lie on the positive, real axis of one of the variables z_k.

If we multiply the function $H^A(z_k)$ in (A. 44) (or the function in (A. 46)) with an arbitrary weight function $\varphi(a_1, a_2, a_3)$ and integrate over the a_k's from zero to infinity, the result is a function that fulfils all the analytic requirements of our functions (provided that the weight behaves in such a way at infinity that the a-integrations converge). As this is a very natural generalization of what is obtained from a straightforward application of perturbation theory, the conjecture that this is a general representation of the functions that fulfil our requirements I, II, and III in the main text is not unreasonable. However, this conjecture is not correct. This follows immediately from the discussion of the singularities given here. As we have shown explicitly that a function obtained in this way has no singularities in some places where singularities can appear according to the general argument, this representation cannot be the most general one. It has been suggested that one obtains a more general function still fulfilling all the analytic requirements by adding to such an expression terms of the form[27]

$$\iint_{-\infty}^{a_3} da_1\, da_2 \int_0^{\infty} da_3\, \varphi(a_k) \iint_{-\infty}^{0} d\alpha_1\, d\alpha_2 \int_1^{\infty} d\alpha_3 \frac{\delta(1-\alpha_1-\alpha_2-\alpha_3)}{z_1\alpha_2\alpha_3 + z_2\alpha_3\alpha_1 + z_3\alpha_1\alpha_2 - a_1\alpha_1 - \alpha_2 a_2 - a_3\alpha_3}, \quad \text{(A. 51)}$$

and the terms obtained from (A. 51) after permutations of indices 1 and 3 and of 2 and 3. The corresponding integrals with the denominators squared can again be expressed in terms of elementary functions. The result of these calculations contains *two* of the three terms in (A. 46) plus an extra term of the form

[27] J. Schwinger, Proc. of the Seventh Annual Rochester Conference, p. IV-32 (1957).

$$\frac{\sqrt{\lambda(z)}}{\Phi} \log \frac{z_1+z_2-z_3+\sqrt{\lambda(z)}}{z_1+z_2-z_3-\sqrt{\lambda(z)}}. \tag{A.52}$$

As we don't have all three terms in (A. 46) here, the calculation leading to (A. 50) cannot be carried through in general. Therefore, these terms have singularities as soon as $\Phi=0$, unless some very intricate cancellations happen between the different terms coming from (B. 9) and its permutations. As $\Phi=0$ has roots in many places inside the domain of holomorphy (cf. the discussion above) it follows that the extra terms (A.51) are not possible as representations of our function for *arbitrary* weight functions $\varphi(a_k)$. Further, in those parts of (A. 51) that have negative values of at least one of the masses a_k, it follows that the "thresholds" (i) discussed above are *complex* quantities. This is in contradiction with the requirement that all singularities whose position depends on only one of the z's, lie on the positive, real axis. Finally, it might also be mentioned that the term exhibited in (A. 52) can be shown to have singularities inside the domain of holomorphy U.

A characteristic feature of the example discussed above is the appearance of non-trivial singularities in the function $H(z)$ in p-space, but of no singularities, except on the cuts, for the function $F(z)$ in x-space. We want to mention that there exists another example from perturbation theory where the situation is reversed. We consider three scalar fields $\Phi_k(x)$ with masses m_k ($k=1, 2, 3$) which interact with the Lagrangian $L_{\text{int}}=\lambda\,\Phi_1(x)\,\Phi_2(x)\,\Phi_3(x)$. The first two terms in an expansion of the field operators in powers of λ are

$$\Phi_k(x)=\Phi_k^{(0)}(x)+\lambda\int \Delta_R(x-x', m_k)\,\Phi_l^{(0)}(x')\,\Phi_m^{(0)}(x')\,dx'+\ldots, \tag{A.53}$$

where

$$(\Box-m_k^2)\,\Delta_R\,(x-x', m_k)=-\delta(x-x') \tag{A.53 a}$$

and

$$\Delta_R(x-x', m_k)=0 \text{ for } x_0-x_0'<0. \tag{A.53 b}$$

The first non-vanishing term in the vacuum expectation value of the three fields is given by

$$\left.\begin{aligned}&\langle 0\,|\,\Phi_1(x)\,\Phi_2(x')\,\Phi_3(x'')\,|\,0\rangle=\frac{\lambda}{(2\pi)^6}\iint dp\,dp'\,e^{ip(x-x')+ip'(x'-x'')}\\ &\times\left[\frac{\delta((p-p')^2+m_2^2)\,\delta(p'^2+m_3^2)}{p^2+m_1^2}\,\Theta(p')\,\Theta(p-p')+\frac{\delta(p^2+m_1^2)\,\delta(p'^2+m_3^2)}{(p-p')^2+m_2^2}\,\Theta(p)\,\Theta(p')\right.\\ &\left.\qquad+\frac{\delta(p^2+m_1^2)\,\delta((p'-p)^2+m_2^2)}{p'^2+m_3^2}\right]\Theta(p)\,\Theta(p'-p).\end{aligned}\right\} \tag{A.54}$$

From this, we can compute the function $H(z)$ and find the simple result

$$H(z) = -\lambda \prod_{k=1}^{3} (m_k^2 - z_k)^{-1}. \tag{A.55}$$

This function is analytic when the variables z_k vary independently in their cut planes. The domain of analyticity for $H(z)$ is in this case exactly the same as the domain of analyticity for $F(z)$ in the previous example.

The direct computation of the function $F(z)$ from (A. 54) is somewhat involved. Instead of doing this calculation we remark that, if we multiply the expression (A. 54) with an arbitrary weight function $\varphi(m_1, m_2, m_3)$ and integrate it over positive values of the masses m_k, the result still fulfils all our requirements about local commutativity and the mass spectrum. Of particular interest is the following special weight.

$$\varphi(m_k) = \prod_{k=1}^{3} \bar{\Delta}\,(-m_k^2, a_k)\, 2\, m_k;\; a_k > 0, \tag{A.56}$$

$$\bar{\Delta}(a, b) = P \frac{1}{(2\pi)^4} \int \frac{dp\, e^{ipx}}{p^2 + b}\bigg|_{x^2 = a} = \frac{1}{8\pi^2} \int_{-\infty}^{+\infty} dw\, e^{-iaw + \frac{ib}{4w}}. \tag{A.56 a}$$

With the aid of the relation

$$P \int_0^{\infty} \frac{da\, \bar{\Delta}(-a, b)}{p^2 + a} = \frac{\pi}{2} \Delta^{(1)}(p^2, b) = \frac{1}{16\pi^2} \int dx\, e^{ipx}\, \delta(x^2 + b), \tag{A.57}$$

we find

$$\left.\begin{aligned} J &= \iiint_0^{\infty} \varphi(m_k)\, dm_1\, dm_2\, dm_3 \langle 0 \,|\, \Phi_1(x)\, \Phi_2(x')\, \Phi_3(x'') \,|\, 0 \rangle \\ &= \frac{\pi\lambda}{2(2\pi)^6} \iint dp\, dp'\, e^{ip(x-x') + ip'(x'-x'')} [\bar{\Delta}((p-p')^2, a_2)\, \bar{\Delta}(p'^2, a_3)\, \Delta^{(1)}(p^2, a_1)\, \Theta(p')\, \Theta(p-p') \\ &\quad + \bar{\Delta}(p^2, a_1)\, \bar{\Delta}(p'^2, a_3)\, \Delta^{(1)}((p-p')^2, a_2)\, \Theta(p)\, \Theta(p') \\ &\quad + \bar{\Delta}(p^2, a_1)\, \bar{\Delta}((p-p')^2, a_2)\, \Delta^{(1)}(p'^2, a_3)\, \Theta(p)\, \Theta(p'-p)]. \end{aligned}\right\} \tag{A.58}$$

With the aid of standard techniques, this integral can be rearranged to read

$$J = \frac{-\lambda}{8} \frac{1}{(2\pi)^7} \iiint_0^1 \frac{d\alpha_1\, d\alpha_2\, d\alpha_3\, \delta(1 - \alpha_1 - \alpha_2 - \alpha_3)}{z_1 \alpha_2 \alpha_3 + z_2 \alpha_3 \alpha_1 + z_3 \alpha_2 \alpha_1 - a_1 \alpha_1 - a_2 \alpha_2 - a_3 \alpha_3}. \tag{A.59}$$

This is exactly the same integral that was studied in (A. 44) and was shown to have non-trivial singularities when not all the z_k's lie in the same half plane. By integrating with a suitable weight function the result obtained from perturbation theory, it is therefore possible to obtain singularities off the cuts also for the function $F(z)$ in x-space.

It is somewhat remarkable that none of these examples obtained from perturbation theory yields a singularity when all points z_k lie in the same half-plane. The assumptions used here about the mass spectrum and local commutativity do not imply such a large analyticity domain. It is an open question for the moment whether or not further general conditions like the unitarity of the S-matrix enlarge the domain of analyticity of our functions or if the large domains found in perturbation theory are merely consequences of the very special interactions studied there. There is also the possibility that these large domains are consequences of the expansions used.

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